PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1990 **331**, 195-244
doi: 10.1098/rsta.1990.0065

Phil. Trans. R. Soc. Lond. A **331**, 195–244 (1990) [195]
Printed in Great Britain

SOLITARY-WAVE SOLUTIONS OF NONLINEAR PROBLEMS

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(*Received* 30 *September* 1988 – *Revised* 3 *October* 1989)

CONTENTS

A general method is presented for the exact treatment of analytical problems that have solutions representing solitary waves. The theoretical framework of the method is developed in abstract first, providing a range of fixed-point theorems and other useful resources. It is largely based on topological concepts, in particular the fixed-point index for compact mappings, and uses a version of positive-operator theory referred to Fréchet spaces. Then three exemplary problems are treated in which steadily propagating waves of permanent form are known to be represented. The first covers a class of one-dimensional model equations that generalizes the classic Korteweg–de Vries equation. The second concerns two-dimensional wave motions in an incompressible but density-stratified heavy fluid. The third problem describes solitary waves on water in a uniform canal.

1. Introduction

The aim of this study is to establish a general technique for the exact treatment of nonlinear problems that have solutions in the form of solitary waves. Such problems arise from a variety of physical contexts, having correspondingly various mathematical formulations, and they have practical importance as representations of observable wave phenomena. As evinced by the proliferation of literature and of conferences related to the subject, interest in solitary waves has grown enormously over the last three decades, having spread to many different branches of applied sciences.

Except in the simplest examples, however, such as the well-known model equation of Korteweg and de Vries whose solitary-wave solutions are known explicitly, the analysis of solitary waves presents difficulties beyond those respective to periodic steady waves in the same systems. The difficulties arise because any solitary wave has an unbounded domain of definition. To a large extent for this reason, the earlier exact theory developed *ad hoc* for particular physical applications was limited to waves of small amplitude (see, for example, Friedrichs & Hyers 1954; Ter-Krikorov 1963; Benjamin 1966). Even some recent work lies in the realm of small-amplitude theory (see, for example, Amick & Kirchgässner 1989; Amick & Turner 1989; Bona & Sachs 1989; Sachs 1990), but interest has also been attached to the provision of global solitary-wave theories. The topological methods to be expounded presently are free from restrictions on wave amplitudes and offer a wide range of prospective applications. The methods, which rely on comparatively coarse topologies, circumvent the lack of compactness that is typically associated with continuous operators acting in Banach spaces of functions defined on unbounded domains. Our approach complements other modern resources that have been applied successfully to problems comparable with those to be considered (e.g. variational techniques, bifurcation theory and methods of concentrated compactness, to which references will be made below).

We shall present our investigation in two stages. First, in §2, a theory is developed in abstract concerning nonlinear operator equations of the form $u = Au$ posed in a Fréchet space X. The account deals particularly with equations having multiple solutions, which are conveniently delimited by means of a specification that the operator A takes a cone $K \subset X$ into itself. The main, subsequently useful results are obtained by index-theoretic arguments. Then, in §§3, 4 and 5, three specific applications of the general theory are presented, establishing the existence and other properties of solitary-wave solutions in physically relevant examples.

The first application (§3) concerns a wide class of one-dimensional model equations for wave propagation in nonlinear dispersive systems. These equations are only approximate, being comparable with the Korteweg–de Vries equation in formal status, but the solitary-wave problem for them is an instructive prototype whose existence theory we outlined in earlier work (Bona *et al.* 1976; Bona 1981 *a*). The problem is recast as an integral equation

$$\varphi(x) = \int_{\mathbb{R}} k(x-y)\, \varphi^2(y)\, dy, \tag{1.1}$$

whose required solution $\varphi:\mathbb{R}\rightarrow\mathbb{R}$ is, like the kernel k, a bounded, positive and even function which decreases monotonically to zero as its argument approaches infinity. The kernel is normalized so that the constant function taking the value 1 everywhere is a solution, as also is the constant function zero, and (1.1), moreover, has periodic solutions with any period greater than a certain number (Benjamin 1974, §5). The solitary-wave theory accordingly has to exclude a large multiplicity of extraneous solutions. The results of our theory are closely allied to those obtained recently by Weinstein (1987) using the method of concentrated compactness developed by Lions (1984*a*, *b*).

The second application (§4) concerns progressive waves of permanent form in density-stratified heavy fluids. The hydrodynamic equations lead to a semi-linear elliptic problem defined on the infinite strip $S=\mathbb{R}\times[0,1]$ (cf. Yih 1980, p. 104; Benjamin 1971, pp. 640–642). A non-zero function $\varphi:S\rightarrow[0,\infty)$ is sought satisfying

$$\left.\begin{aligned}\Delta\varphi+\varphi f(y,\varphi)=0 \quad &\text{in} \quad \mathbb{R}\times(0,1),\\ \varphi(x,0)=0, \quad \varphi(x,1)=0, \quad &\text{for all} \quad x\in\mathbb{R},\end{aligned}\right\} \tag{1.2}$$

and vanishing in the limits $x\rightarrow\pm\infty$. Here $f:[0,1]\times[0,\infty)\rightarrow[0,\infty)$ is a function complying with various conditions, and the boundary-value problem (1.2) has a second x-independent solution as well as the zero solution. This problem too has a continuum of periodic solutions, and after recasting it as an integral equation we treat it in much the same way as the first, prototypal problem. Our results add to the already extensive literature in this area (particularly Amick 1984), which is reviewed in §4.

Finally, in §5, we deal with an integral equation derived exactly from the nonlinear problem for solitary water waves in a uniform horizontal canal. This is the original context in which solitary waves arose, and the approximate analyses by Rayleigh, Boussinesq and Korteweg & de Vries in the last century are the classics of our subject (cf. Benjamin (1974) Bona (1980*b*) and Newell (1985) for historical commentary). In the considered formulation as an integral equation, the problem has no non-zero constant solution; and for this reason, notwithstanding a seemingly more complicated integral operator A, its treatment by the present means is simpler than the preceding two examples. A global though non-constructive existence theory is readily completed, complementing the treatment by quite different means that has been given by Amick & Toland (1981*a*, *b*).

As exemplified by these three problems, solitary waves are non-constant, non-periodic functions whose support is unbounded, and the operator equations for them are invariant under translations in the unbounded direction of their domain of definition S. These features are responsible for the special difficulties of solitary-wave theory, notably because they tend to exclude a theory based upon the use of various fixed-point theorems of Schauder–Tychonov type, index theorems and other powerful resources applicable to *compact* operator equations posed in Banach spaces. If, as may be quite reasonable otherwise, a solitary wave is attributed to some Banach space Y of functions defined on S (e.g. $C_b(S)$ or $L^2(S)$), being considered accordingly as a fixed point of the operator A in Y, the operator will generally not be compact in the required sense (i.e. it will not transform bounded sets in Y into sets that are relatively compact, having compact closures). In a check on the possibility of A being compact, equicontinuity of the general image set may be confirmable, but the further necessary condition that it be equi-small at infinity (see Yosida 1974, ch. 10) will turn out not to be fulfilled.

On the other hand, certain classes of functions on S considered as Fréchet spaces, so being endowed with a comparatively coarse topology, are freed from the latter condition for the relative compactness of a subset. We adopt the Fréchet-space formulation of our general theory for this crucial advantage, enabling us to specify a useful sense in which the relevant operators are compact. Needless to say, this strategy has precedents: for instance, it was used by Weyl (1942) and Hastings (1970) in applications to boundary-layer theory, by Bushell (1972) who thereby brought the Fréchet-space version of Tychonov's fixed-point theorem to bear on a problem of rotating fluids, by Taylor (1967) and by Krasnosel'skii *et al.* (1973).

Offsetting this advantage, special difficulties need to be overcome in working with a non-normable Fréchet space X. For the most part they are due to the inherent property of such spaces that every open subset is unbounded (see §2 for a résumé of such basic facts about Fréchet spaces). This fact severely restricts the class of operators A that map subsets of X compactly into X, and so limits the availability of an index theory. In other words, with $\mathscr{B}_r$ denoting any metric ball of radius r in X, the useful condition that $A(\mathscr{B}_r)$ be a relatively compact subset of X for a suitable $r > 0$ excludes the operators typically presented in applications, including those to be considered in §§3–5.

The difficulty is obviated in our approach by appropriately narrowing the domain of A, specifically to some cone $K \subset X$ in which the solitary-wave solution is recognized *a priori* to be an element. Each of the cones K chosen in the applications is sufficiently narrow for its intersection with any suitable ball $\mathscr{B}_r$ to be bounded, and it is consequently mapped by the respective A into a relatively compact subset of X. This property enables us to adapt to the Fréchet-space problem various parts of the theory of positive operators that has been developed for Banach spaces by Krasnosel'skii (1964*a*) and others (see, for example, Amann 1976; Deimling 1985, ch. 5). In particular, we establish versions of the fixed-point theorems for nonlinear operators that 'expand' or 'compress' a cone (Krasnosel'skii 1964*a*, ch. 4). The first of these theorems, coupled with an argument in terms of topological index which discriminates between the solitary-wave solution and the non-zero trivial solution, is applied in §§3 and 4 to the problems (1.1) and (1.2). The second of the theorems serves in §5 to treat the problem of solitary water waves.

It should be acknowledged that an alternative, indirect approach to existence theory for solitary waves consists in first proving the existence of steady periodic waves with arbitrarily large period, and then obtaining the solitary wave in the limit as the period tends to infinity. This approach, which poses various technical difficulties of its own, has been used by us previously for particular problems (Bona *et al.* 1976; Bona *et al.* 1983), and it has been used very effectively by Amick & Toland (1981 *a*, *b*) in their theory of the surface solitary wave. A distinctive feature of our method is that solitary waves are established without recourse to an existence theory for periodic waves. An incidental aspect of the theory covered in §2, however, is an interesting connection in topological terms between steady periodic waves and solitary waves. This sidelight on our analysis reveals in particular a simple criterion whereby a solitary wave is guaranteed to be the limit of steady periodic waves.

2. Positive-operator theory for Fréchet spaces

Our aim is to exhibit various useful properties of continuous operators whose domain constitutes a subset of a Fréchet space. In its general description, a Fréchet space is a metrizable and complete, locally-convex, linear topological space. Here linear spaces will always be understood as vector spaces over the field of real numbers. Any Fréchet space X can be described in the following specific terms (cf. Rudin 1973, ch. 1; Treves 1967, ch. 10). On X a sequence $\{p_n\}_{n=1}^{\infty}$ of semi-norms can be defined which is increasing (i.e. $p_{n+1}(x) \geqslant p_n(x)$ for every $x \in X$ and every $n = 1, 2, 3, \ldots$) and is such that the formula

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x-y)}{1+p_j(x-y)}, \quad \text{for} \quad x, y \in X, \tag{2.1}$$

provides a metric generating a topology that coincides with the original topology on X. Such a choice of metric occurs naturally for many of the specific Fréchet spaces having present interest. Accordingly, it will henceforth be assumed that any Fréchet space in question is equipped with a countable family of semi-norms as above, and the metric will be presumed to have the form (2.1). This situation will be recalled by the following phrase: X is a Fréchet space with generating family of semi-norms $\{p_n\}_{n=1}^{\infty}$.

For any Fréchet space X with generating family of semi-norms $\{p_n\}_{n=1}^{\infty}$, write

$$\left.\begin{aligned} \mathscr{B}_r &= \{x \in X : d(x,0) < r\} \\ \text{and} \qquad \mathscr{B}_r^j &= \{x \in X : p_j(x) < r\}, \end{aligned}\right\} \tag{2.2}$$

where j is any positive integer. Note that $\mathscr{B}_1 = X$, in consequence of (2.1).

In general, a set B in a topological linear space X is said to be *bounded* if, for any neighbourhood U of 0 in X, there is a $\lambda > 0$ such that $\lambda B \subseteq U$. In the case of a Fréchet space with metric d as in (2.1), a set B in X is bounded if and only if corresponding to each positive integer j there is an $R > 0$ such that $B \subseteq \mathscr{B}_R^j$. If $r > 0$, then $\mathscr{B}_r$ is usually not bounded. In fact, if X is a Fréchet space that has any bounded open set, X is necessarily a Banach space (Rudin 1973, ch. 1).

To complete our summary of needed theory, we first note how a Fréchet space can be partially ordered by a cone and we introduce certain definitions of compactness for operators that map a cone into itself. Standard elements of topological index theory are then reviewed in §2.2 and several useful consequences are derived. The notion of r-domination is introduced in §2.3, being used to relate the fixed points of sequences of operators (proposition 2.5) and so illuminate the general connection between solitary-wave and periodic-wave solutions of nonlinear problems. Finally, in §2.4, certain fixed-point theorems are proven that will be applied in subsequent sections to the treatment of particular problems having solitary-wave solutions.

2.1. *Cones in Fréchet spaces*

It will be of importance in the sequel to consider subsets K of X that are called cones. A closed subset K of a linear space X is a *cone* if

$$\left.\begin{aligned} &\text{(i)} \quad K+K = \{u+v : u, v \in K\} \subseteq K, \\ &\text{(ii)} \quad \lambda K = \{\lambda u : u \in K\} \subseteq K \quad \text{for all} \quad \lambda \geqslant 0, \\ &\text{(iii)} \quad K \cap (-K) = \{0\}. \end{aligned}\right\} \tag{2.3}$$

Note that (i) and (ii) imply K to be a convex subset of X. Any cone K in X defines a partial ordering according to the prescription

$$x \prec y \quad \text{if and only if} \quad y - x \in K.$$

A cone K in a Fréchet space X is said to be *normal* if there is a family of semi-norms $\{p_j\}_{j=1}^{\infty}$ generating the metric topology on X such that, if $x, y \in K$ with $x \prec y$, then

$$p_j(x) \leqslant p_j(y)$$

for all $j \geqslant 1$. Another useful class of cones are those we shall describe as p_1-bounded. A cone K in X is designated p_1-bounded if there is a sequence of positive numbers $\{a(n)\}_{n=1}^{\infty}$ such that

$$p_n(x) \leqslant a(n)\, p_1(x)$$

for all x in K and positive integers n.

Let K be a cone in a Fréchet space X. A mapping A defined (at least) on K and with range in X is called a *positive operator* if $A(K) \subseteq K$. The class of operators on X that will be the primary focus of attention presently are positive, continuous and compact in the following sense. A positive operator A is K-compact if $A(K \cap \mathscr{B}_r)$ has compact closure, for each r in $[0, 1)$. Another useful notion of compactness is embodied in the following definition. For a given $j \geqslant 1$, a positive operator A is p_j-compact if, for any positive $R > 0$, any sequence in $A(K \cap \mathscr{B}_R^j)$ has a subsequence that converges in the semi-norm p_j to an element of K. A positive operator is said to be p-compact if it is p_j-compact for all $j \geqslant 1$.

It will appear that K-compactness is a useful concept in the important task of defining a topological index theory for an operator A. Whereas p-compactness is easily established for the operators arising in the applications to solitary-wave theory, in general p-compactness of an operator A does not imply its K-compactness, although it is always the case that a K-compact operator is p-compact. However, these notions are equivalent in the presence of one additional assumption, as we now show.

LEMMA 2.1. *Let X be a Fréchet space with generating family of semi-norms $\{p_n\}_{n=1}^{\infty}$. Let K be a cone in X and A a p-compact positive operator on X. Suppose that, for each r in $[0, 1)$, $K \cap \mathscr{B}_r$ is a bounded set. Then A is a K-compact.*

Proof. Fix an r in $[0, 1)$. As the topology is metrizable, it suffices to show that the closure of $A(K \cap \mathscr{B}_r)$ is sequentially compact. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence from $A(K \cap \mathscr{B}_r)$. As $K \cap \mathscr{B}_r$ is bounded, $p_j(K \cap \mathscr{B}_r)$ is for each j a bounded set in $[0, \infty)$; and because of this property and the p-compactness of A, a Cantor diagonalization process yields a subsequence $\{y_k\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and elements z_j in K such that $p_j(y_k - z_j) \to 0$ as $k \to \infty$, for each j. As the semi-norms $\{p_j\}_{j=1}^{\infty}$ are increasing with j, it follows readily that $p_i(z_l - z_j) = 0$ provided that $i \leqslant j, l$. Hence, if $k \leqslant l$, one has

$$d(z_l, z_k) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(z_l - z_k)}{1 + p_j(z_l - z_k)} \leqslant \sum_{j \geqslant k} 2^{-j} = 2^{-k+1}.$$

This inequality shows that $\{z_k\}_{k=1}^{\infty}$ is a Cauchy sequence in K. Let z be the limit of $\{z_k\}_{k=1}^{\infty}$. As K is closed, z lies in K and plainly $p_j(z_j - z) = 0$ for all j, and hence $p_j(y_k - z) \to 0$ as $k \to \infty$ for each j. The latter conclusion is exactly equivalent to the stipulation that $y_k \to z$ in the metric d. Thus $A(K \cap \mathscr{B}_r)$ is sequentially compact and, as r was an arbitrary number in $[0, 1)$, the proof is complete. □

2.2 *The fixed-point index*

An indispensible tool, used both for further developments in the present section and subsequently, is the fixed-point index for compact operators. A summary of the basic facts regarding this theory is given here, together with a few simple consequences. A detailed development of the fixed-point index for arbitrary absolute neighbourhood retracts, which we use in the context of Fréchet spaces, is to be found in the work of Granas (1972).

Let X be a Fréchet space. The collection of all convex subsets of X is denoted by $\mathscr{C}$, or $\mathscr{C}(X)$ when X is not obvious from the context. A triple (C, A, U) is called admissible if

(i) $C \in \mathscr{C}$,

(ii) $U \subseteq C$ is open in the relative topology on C,

(iii) $A: C \to C$ is continuous and $A(U)$ is a subset of a compact set in C, and

(iv) A has no fixed point on ∂U, the boundary of the open set U in the relative topology on C.

The set of all admissible triples is denoted by $\mathscr{A}$, or by $\mathscr{A}(X)$ if X is not evident from the context. Granas's work cited previously implies the existence of an integer-valued function i defined on $\mathscr{A}$ that satisfies the basic axioms of a fixed-point index (cf. Dold 1965). We record as follows those axioms that find direct use later, together with other elementary properties that follow from Granas's method of constructing the fixed-point index and from the classic Leray–Schauder index theory (cf. Leray & Schauder 1934; Nagumo 1951).

I. (*Localization*) If $(C, A, U) \in \mathscr{A}$ and $B: C \to C$ satisfies (iii) above together with $B(x) = A(x)$ for all $x \in \bar{U}$ (the closure of U in C), then

$$i(C, A, U) = i(C, B, U).$$

II. (*Homotopy invariance*) For $C \in \mathscr{C}$ and a continuous mapping $H: C \times [0, 1] \to C$, let $A_t(\cdot) = H(\cdot, t)$. If there is an open set U, and a compact subset G of C for which $A_t(U) \subseteq G$ and $(C, A_t, U) \in \mathscr{A}$ for all t in $[0, 1]$, then

$$i(G, A_0, U) = i(G, A_1, U).$$

(Note that the set G must be chosen independently of t.) A mapping $H: C \times [0, 1] \to C$ having these properties will be called an admissible homotopy. The operators A_0 and A_1 will be called homotopic.

III. (*Additivity*) If $(C, A, U) \in \mathscr{A}$ and $U_1, \ldots, U_n$ is a collection of mutually disjoint open subsets of U such that $A(x) \neq x$ for all x in $U \backslash \bigcup_{j=1}^{n} U_j$, then

$$i(C, A, U) = \sum_{j=1}^{n} i(C, A, U_i).$$

IV. (*Fixed-point property*) If $(C, A, U) \in \mathscr{A}$ and $i(C, A, U) \neq 0$, then A has at least one fixed point in U.

V. (*Commutativity*) If $C_1, C_2 \in \mathscr{C}$ and $A: C_1 \to C_2$, $B: C_2 \to C_1$ are continuous maps for which $(C_1, B \circ A, U) \in \mathscr{A}$, then $(C_2, A \circ B, B^{-1}(U)) \in \mathscr{A}$ and

$$i(C_1, B \circ A, U) = i(C_2, A \circ B, B^{-1}(U)).$$

VI. (*Index of constant maps*) If $(C, A, U) \in \mathscr{A}$ where, for some $u \in C$, $A(x) = u$ for each $x \in C$, then

$$i(C, A, U) = \begin{cases} 0 & \text{if } u \notin U, \\ 1 & \text{if } u \in U. \end{cases}$$

We record here two simple consequences of properties I–VI which will find use later in this section. Proposition 2.2 was implied in Krasnosel'skii's monograph (1964*b*, p. 166), although it was neither stated explicitly nor proved there. Its demonstration in a Fréchet-space setting was given first by Bona (1972), later by Fitzpatrick & Petryshyn (1976) and Krasnosel'skii & Zabreiko (1984). Proposition 2.3 is a standard implication of index theory. Short proofs of both results are included for convenience of the reader.

Proposition 2.2. *Let X be a Fréchet space and let p be a continuous semi-norm on X. Let K be a cone in X and A a positive and continuous operator on K. Let U be a relatively open subset of K such that $p(U)$ is bounded. Suppose (K, A, U) is admissible and there is a $u \in K$ with $p(u) > 0$ such that*

$$x - Ax \neq au \tag{2.4}$$

for all real numbers $a \geqslant 0$ and $x \in \partial U$. Then $i(K, A, U) = 0$.

Proof. Consider the homotopy $H: K \times [0, 1] \to K$ given by

$$A_t(x) = H(x, t) = A(x) + tau.$$

The constant $a > 0$ will be chosen presently. Because of (2.4) it is apparent that (K, A_t, U) is admissible. Further, for any t in $[0, 1]$, we have $A_t(U) \subseteq \overline{A(U)} + V$, where V is the line segment in X joining 0 and au. As both $\overline{A(U)}$ and V are compact, so also is their sum and thus H is an admissible homotopy. The homotopy invariance of the fixed-point index therefore implies

$$i(K, A, U) = i(K, A + au, U).$$

For large enough values of a, however, it must be expected that $A(x) + au \notin U$, for all $x \in U$. Presuming the truth of this supposition, we can infer from property IV of the fixed-point index that $i(K, A + au, U) = 0$. The proposition will thus be established when it is verified that $\{A(U) + au\} \cap U = \varnothing$ for some value of $a > 0$. To this end, note that by our assumption there is a constant e such that

$$p(x) \leqslant e$$

for all $x \in U$, and by the triangle inequality applied to p

$$p(A(x) + au) \geqslant ap(u) - p(A(x)).$$

Because $\overline{A(U)}$ is compact and p is continuous, there is a $c > 0$ such that $p(\overline{A(U)}) \subseteq [0, c]$. Thus, for any $x \in U$, we have

$$p(A(x) + au) \geqslant ap(u) - c.$$

Hence $A(U) + au$ and U are disjoint provided $a > (c + e)/p(u)$. □

Proposition 2.3. *Let X be a Fréchet space and let $(C, A, U) \in \mathscr{A}$. Suppose there is an open set W in X and a continuous map $B: C \to C$ satisfying the following properties:*

(a) $\lambda W \subseteq W$ for $|\lambda| \leqslant 1$;

(b) $W \cap (I - A)(\partial U) = \varnothing$, where I denotes the identity mapping of C;

(c) $A(x) - B(x) \in W$ for each $x \in \bar{U}$ and $B(U)$ is contained in a compact subset of C.

It then follows that $(C, B, U) \in \mathscr{A}$ and $i(C, A, U) = i(C, B, U)$.

Proof. This result is an immediate consequence of property (II) once it is appreciated that $H: C \times [0, 1] \to C$ given by

$$H(x, t) = tA(x) + (1 - t)B(x)$$

is an admissible homotopy. For any $t \in [0, 1]$ and $x \in U$, we have $H(x, t) \in V_1 + V_2$, where

$$V_1 = \{\lambda v : 0 \leqslant \lambda \leqslant 1, v \in \overline{A(U)}\}, \quad V_2 = \{\lambda w : 0 \leqslant \lambda \leqslant 1, w \in \overline{B(U)}\};$$

and because both V_1 and V_2 are compact, so is their sum. Finally, if $(x,t)\in\partial U\times[0,1]$ and $x = H(x,t)$, it follows that
$$x-A(x) = (t-1)\,(A(x)-B(x))\in W,$$
which contradicts our hypotheses. This contradiction shows at once that $(C,B,U)\in\mathscr{A}$ and that H is admissible on U. □

Proposition 2.2 means that, whenever $A(\partial U)$ misses a ray emanating from the origin, the index of A relative to U must be zero. Proposition 2.3 means that maps uniformly close to each other on a set U necessarily have the same index there.

2.3. *r-domination*

The technical notion of r-domination is introduced here and its usefulness in relating fixed-point indices is indicated. The notion plays a crucial role in our solution of the problems considered in §§3 and 4.

The setting is, as before, a Fréchet space X. Suppose that D and C are convex subsets of X. The set D is said to r-dominate C if there are continuous maps $r:D\to C$ and $s:C\to D$ such that $r\circ s = I_C$, the identity mapping of C.

LEMMA 2.4. *Let $C_1, C_2\in\mathscr{C}$ and suppose that C_1 is r-dominated by C_2. Let (C_1,B,V) be admissible and $A:C_2\to C_2$ a continuous map. Suppose that A and sBr are related by an admissible homotopy $H:C_1\times[0,1]\to C_1$ on the open set $r^{-1}(V)$. Then $(C_2,A,r^{-1}(V))$ is admissible and $i(C_1,B,V) = i(C_2,A,r^{-1}(V))$.*

Proof. As $rs = I_{C_1}$, we have
$$i(C_1,B,V) = i(C_1,rsB,V) = i(C_2,sBr,r^{-1}(V)),$$
because of the commutative property of the fixed-point index. As sBr is admissibly homotopic to A on $r^{-1}(V)$, by hypothesis, the homotopy invariance of the fixed-point index yields the desired conclusion. □

A fixed point x_0 of an operator A defined on a set C is *isolated* if there is a relatively open set V in C containing x_0 with
$$\{x_0\} = \bar{V}\cap\{x\in C: x = A(x)\}.$$
Let $(C,A,U)\in\mathscr{A}$ and let x_0 be an isolated fixed point of A lying in U. The index of A at x_0, written $i(A,x_0)$, is
$$i(A,x_0) = i(C,A,V), \tag{2.5}$$
where $V\subseteq U$ is an open set containing x_0 but no other fixed point of A. This concept is well defined, as appears from the additive property of the fixed-point index.

The next result refers specifically to the solitary-wave theory that is the focus of interest here. It will appear in §§3, 4 and 5 that the existence of solitary-wave solutions in various systems may be shown to depend on whether a nonlinear operator has a fixed point in a cone of non-negative functions in a suitable function space. The latter issue may be decided in the affirmative by use of the techniques set out in this section. In carrying out this programme, an interesting by-product emerges, which now will be explained.

Systems that support solitary waves appear always to possess solutions representing spatially periodic wavetrains of permanent form. Such wavetrains were first noticed by Korteweg & de Vries (1895) in their treatment of long water waves, and were named by them *cnoidal* waves.

Our next proposition gives conditions under which the existence of a solitary-wave solution implies existence of periodic permanent-wave solutions. Moreover, it will also follow that these cnoidal-wave solutions converge to the solitary wave, whose existence was presumed, in the limit of indefinitely large period. The convergence of periodic wavetrains of permanent form to a solitary wave was also first observed by Korteweg & de Vries. Other examples of this phenomenon have been demonstrated in Amick & Toland (1981 *b*), Beale (1977), Bona *et al.* (1983) and Turner (1981). In the special context of the Korteweg–de Vries equation, it has been shown that all solutions defined on the entire real line which are appropriately evanescent at infinity, whether or not they are steadily propagating waves of permanent form, arise as limits of periodic wavetrains (see Bona 1981 *b*).

Guided by the examples worked out in §§3, 4 and 5, we assume X to be a Fréchet space with generating family of semi-norms $\{p_j\}_{j=1}^{\infty}$. Let C and each C_j in the sequence $\{C_j\}_{j=1}^{\infty}$ be closed convex subsets of X, and suppose that C r-dominates C_j for all j. Denote by $r_j: C \to C_j$ and $s_j: C_j \to C$ continuous maps such that $r_j \circ s_j = I_j$, the identity map on the set C_j, for each $j = 1, 2, \ldots$. Let A and $\{B_j\}_{j=1}^{\infty}$ be continuous operators, $A: C \to C$ and $B_j: C_j \to C_j$ for $j = 1, 2, \ldots$. We set out the following conditions regarding these operators.

(A1) For any $R < 1$, $d(Ax, s_j B_j r_j x) \to 0$ as $j \to \infty$, uniformly for x in $C \cap \mathscr{B}_R$.

(A2) For any $R < 1$, $A(C \cap \mathscr{B}_R)$ is contained in a compact subset of C and, for each j, $B_j(C_j \cap \mathscr{B}_R)$ is contained in a compact subset of C_j.

(A3) As $j \to \infty$, $d(r_j x, x) \to 0$ uniformly for x in C.

(Note that (A1), (A2) and the triangle inequality imply that, for $R < 1$, A is uniformly approximated by $B_j r_j$ on the set $C \cap \mathscr{B}_r$.)

We wish to answer the following question. Suppose that $\phi \in C$ is a fixed point of the operator A. When is there a sequence $\{\phi_j\}_{j=1}^{\infty}$ converging to ϕ, with each ϕ_j a fixed point of B_j for all j sufficiently large?

Proposition 2.5. *With notation and assumptions as just presented, suppose ϕ to be an isolated fixed point of A in C with non-zero index. Then there exists a sequence $\{\phi_j\}_{j=1}^{\infty}$ in X that converges to ϕ and is such that $\phi_j = B_j \phi_j$ for all j sufficiently large.*

Proof. As ϕ is an isolated fixed point of A in C, the definition (2.5) of its index $i(A, \phi)$ is recovered by every (relatively) open set V that contains ϕ and is itself contained in $\overline{C \cap \mathscr{B}_\epsilon(\phi)}$, where ϵ is some fixed positive number which is chosen to make $d(0, \phi) + \epsilon < 1$. For $n = 1, 2, \ldots$, define $V_n = C_n \cap \mathscr{B}_{\frac{1}{2}\epsilon}(r_n \phi)$ and consider the relatively open set $U_n = r_n^{-1}(V_n)$ in C. We note that $\phi \in U_n$ for all n and, because U_n is open and so disjoint from its relative boundary ∂U_n in C, therefore $\phi \notin \partial U_n$ for all n. If $v \in U_n$, we have

$$d(v, \phi) \leqslant d(v, r_n v) + d(r_n v, r_n \phi) + d(r_n \phi, \phi),$$

where the second term on the right-hand side is less than $\frac{1}{2}\epsilon$ by the definition of V_n, and according to condition (A3) the first and third terms can be made as small as desired by taking n large enough. Thus an integer N exists such that

$$U_n \subset C \cap \mathscr{B}_\epsilon(\phi) \quad \text{if} \quad n \geqslant N, \tag{2.6}$$

and accordingly the assumption that $i(A, \phi) \neq 0$ implies that

$$i(C, A, U_n) \neq 0 \quad \text{if} \quad n \geqslant N. \tag{2.7}$$

Take any $y \in \partial U_n$ and let $x = r_n y \in r_n(\partial U_n) \subset \partial V_n$ in the present circumstances. It then follows that

$$\tfrac{1}{2}\epsilon = d(r_n \phi, x) \leqslant d(r_n \phi, \phi) + d(\phi, y) + d(y, x),$$

where the third term on the right-hand side is $d(y, r_n y)$. According to condition (A3), this term and the first can be made arbitrarily small, uniformly on C, by taking n large enough. Hence, subject to a revision of the choice of N if required, it can be concluded that, for $n \geqslant N$,

$$d(\phi, y) > \tfrac{1}{4}\epsilon \quad \text{for all} \quad y \in \partial U_n. \tag{2.8}$$

Together (2.6) and (2.8) imply that, for $n \geqslant N$,

$$\partial U_n \subseteq \overline{C \cap \mathscr{B}_\epsilon(\phi)} \backslash \mathscr{B}_{\frac{1}{4}\epsilon}(\phi) = Y, \quad \text{say},$$

and the image $A(Y)$ of this closed set is contained in a compact subset of C. Moreover, Y contains no fixed point of A, and so a number $\delta > 0$ exists such that $d(y, Ay) \geqslant \delta$ for all $y \in Y$. It therefore follows that, for all $n \geqslant N$,

$$d(y, Ay) \geqslant \delta \quad \text{for all} \quad y \in \partial U_n.$$

(Note that δ depends on ϵ, which has been prescribed, but in view of (2.6) and (2.8) δ is independent of $n \geqslant N$.) On the other hand, condition (A1) ensures that $d(Ax, s_n B_n r_n x) < \frac{1}{2}\delta$ for all $x \in C \cap \mathscr{B}_R$ (with any $R < 1$) if n is large enough. Thus, subject to another possible revision of the choice of N, we have that $W = \mathscr{B}_{\frac{1}{2}\delta}(0)$ satisfies the conditions of Proposition 2.3 applied to a comparison between A and $s_n B_n r_n$ on U_n for each $n \geqslant N$. According to Proposition 2.3 and the commutative property V of the fixed-point index, it follows that, for $n = N, N+1, \ldots,$

$$i(C, A, U_n) = i(C, s_n B_n r_n, U_n) = i(C, s_n B_n r_n, r_n^{-1}(V_n)) = i(C_n, B_n, V_n), \tag{2.9}$$

whence (2.7) and property IV of the fixed-point index guarantee that, for this range of n, there is a fixed point ψ_n of $s_n B_n r_n$ in U_n. For $n = N, N+1, \ldots$, let $\phi_n = r_n \psi_n$ so that $B_n \phi_n = \phi_n$ because $r_n s_n$ is the identity map on C_n.

(Note incidentally that the existence of the fixed point ϕ_n also follows from (2.7) and (2.9). This fact provides another, possibly interesting sidelight on the connection between solitary-wave and periodic-wave problems; but it adds nothing to the conclusion of our main argument.)

To complete the proof, we proceed to show that the sequence $\{\phi_n\}_{n=N}^{\infty}$ converges to ϕ in the metric d as $n \to \infty$. First it is noted that because the ψ_n all lie in $C \cap \mathscr{B}_\epsilon(\phi)$, and because $d(\phi, 0) + \epsilon < 1$, condition (A1) implies that

$$d(A\psi_n, \psi_n) = d(A\psi_n, s_n B_n r_n \psi_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.10}$$

But, by condition (A2), $A(C \cap \mathscr{B}_\epsilon(\phi))$ is a subset of a compact set in C, and so either $\{\psi_n\}_{n=N}^{\infty}$ or a subsequence is such that

$$d(A\psi_n, v) \to 0 \quad \text{as} \quad n \to \infty \tag{2.11}$$

for some $v \in C$. The properties (2.10) and (2.11) evidently imply that $\psi_n \to v$ as $n \to \infty$, whence the continuity of A implies that $A\psi_n \to Av$ as $n \to \infty$ and accordingly (2.11) shows that $Av = v$. As ϕ is known to be the only fixed point of A in $\overline{C \cap \mathscr{B}_\epsilon(\phi)}$, it follows that $v = \phi$. As any subsequence of $\{\psi_n\}_{n=N}^{\infty}$ may thus be inferred to possess a further subsequence converging to ϕ, it follows that the entire sequence $\{\psi_n\}_{n=N}^{\infty}$ converges to ϕ.

Finally, we consider

$$d(\phi, \phi_n) \leqslant d(\phi, \psi_n) + d(\psi_n, \phi_n) = d(\phi, \psi_n) + d(\psi_n, r_n \psi_n).$$

The first term on the right-hand side has just been shown to vanish in the limit as $n \to \infty$, and the second term also vanishes in this limit by virtue of condition (A3). Thus the proof is complete. □

The useful implications of this proposition become immediately apparent in specific examples of solitary waves, where the C_n can be chosen as cones of positive periodic functions with period proportional to n. As the foregoing proof indicates, fixed points ϕ_n corresponding to periodic travelling waves of permanent form are generally guaranteed only if n is large enough. This conclusion of the abstract arguments is to be expected, however, for previous studies of particular systems have commonly shown that periodic waves with the same speed as a solitary wave are possible only for large enough wavelengths (cf. Benjamin 1966, 1974, §5). We shall not labour the implications of the present general result in the context of the examples treated later; but the reader will readily appreciate how in every case a direct existence theory for a solitary wave, based on our Fréchet-space methods, automatically subsumes an account of periodic waves that necessarily converge to the solitary wave in the limit as their wavelength is increased without bound.

2.4. *Fixed-point theorems*

In this subsection we shall extend, from their original Banach-space setting to a Fréchet-space setting, certain theorems due to Krasnosel'skii (1964*a*, ch. 4) concerning non-trivial fixed points of positive operators. The extended versions of these theorems are exactly what is needed to establish the existence of solitary waves in the various systems to be treated in subsequent sections of this paper. Our proofs rely on the fixed-point index introduced above, following closely the proofs in Bona (1972) and Fitzpatrick & Petryshyn (1976). The idea of using the fixed-point index to prove these theorems, rather than Krasnosel'skii's method based on *ad hoc* constructions and the Schauder fixed-point theorem, appeared first in Benjamin (1971, Appendix 1). A recent exposition of these results in a Banach-space setting is available in Deimling (1985, ch. 5), and an alternative approach to the generalization of Krasnosel'skii's theorems may be found in Isac (1984). The extra information obtained from the proofs by means of index theory, namely that the fixed-point index of the operator in question is non-zero relative to a certain conical segment, is crucial to our applications.

Two lemmas are stated and proved, from which the fixed-point theorems readily follow. To cover the various applications in §3–5, each of these lemmas is presented with alternative assumptions, the second assumption being weaker than the first. It is assumed throughout that K is a cone in a Fréchet space with generating family of semi-norms $\{p_j\}_{j=1}^{\infty}$ and the standard metric d as in (2.1), and that $A: K \to K$ is continuous and K-compact.

Lemma 2.6. *Suppose that* $0 < \rho < 1$ *and that either*

$$Ax - x \notin K \quad \textit{for all} \quad x \in K \cap \partial\mathscr{B}_\rho, \tag{2.12}$$

or

$$tAx \neq x \quad \textit{for all} \quad x \in K \cap \partial\mathscr{B}_\rho \quad \textit{and all} \quad t \in [0, 1]. \tag{2.13}$$

Then, with K_ρ *denoting* $K \cap \mathscr{B}_\rho$, *the triple* (K, A, K_ρ) *is admissible and*

$$\mathrm{i}(K, A, K_\rho) = 1.$$

Proof. It suffices to assume condition (2.13) because condition (2.12) implies (2.13). Clearly (K, A, K_ρ) is admissible as (2.13) ensures that A has no fixed point on the relative boundary of K_ρ. Consider the mapping $H(x, t) = tAx$ defined on K, and inquire whether it forms an admissible homotopy on the set K_ρ. As A is continuous and K-compact and $\rho < 1$, the compactness and continuity requirements for H are obviously satisfied. It remains to show that H has no fixed point on the relative boundary $K \cap \partial\mathscr{B}_\rho$. If there were an $x_0 \in K \cap \partial\mathscr{B}_\rho$ and a $t_0 \in [0, 1]$ such that $x_0 = H(x_0, t_0) = t_0 Ax_0$, it would contradict (2.13). Thus A is seen to be homotopic to the constant map B such that $Bx = 0$ for all $x \in K$ on K_ρ. It hence follows by properties II and VI of the fixed-point index that $i(K, A, K_\rho) = 1$. □

Lemma 2.7. *Suppose that* $0 < \rho < 1$ *and that either*

$$x - Ax \notin K \quad \textit{if} \quad x \in K \cap \partial\mathscr{B}_\rho, \tag{2.14}$$

or

there exists $u^* \in K$ *satisfying* $p_1(u^*) > 0$ *such that* $x - Ax \neq au^*$ *for all* $x \in K \cap \partial\mathscr{B}_\rho$ *and* $a \geqslant 0$. (2.15)

Then (K, A, K_ρ) *is admissible and*

$$i(K, A, K_\rho) = 0.$$

Proof. It suffices to assume condition (2.15) because condition (2.14) implies (2.15). Then the lemma is just a special instance of proposition 2.2. □

We can now present the needed version of the expanded-cone theorem (cf. Krasnosel'skii 1964*a*, theorem 4.12).

Theorem 2.8. *Let K be a cone in a Fréchet space and let $A: K \to K$ be continuous and K-compact. Suppose that either* (2.12) *or* (2.13) *holds for an r satisfying* $0 < r < 1$ *and that either* (2.14) *or* (2.15) *holds for an R satisfying* $r < R < 1$. *Then A has at least one fixed point in the set*

$$K_r^R = \{x \in K : r < d(x, 0) < R\},$$

and furthermore

$$i(K, A, K_r^R) = -1.$$

Proof. By lemmas 2.6 and 2.7 we know that $i(K, A, K_r) = 1$ and $i(K, A, K_R) = 0$. By property III of the fixed-point index, it follows that $i(K, A, K_r^R) = -1$, and by property IV it follows that A has a fixed point in K_r^R. □

The following compressed-cone theorem (cf. Krasnosel'skii 1964*a*, theorem 4.14) can be proved in similar fashion.

Theorem 2.9. *Let K and A be as in theorem* 2.8. *Taking* $0 < r < R < 1$, *suppose that either* (2.14) *or* (2.15) *holds respective to r, and that either* (2.12) *or* (2.13) *holds respective to R. Then A has at least one fixed point in K_r^R, and furthermore*

$$i(K, A, K_r^R) = +1.$$ □

It is worth remarking that versions of both the compressed-cone and expanded-cone theorems can be formulated in terms of the Fréchet derivative of the positive operator A at 0 and the asymptotic derivative of A at infinity. This type of theorem was formulated and used in an earlier version of §4 (Bose 1974; see also Fitzpatrick & Petryshyn 1976). However, we have found the present theory more easily applicable and so forego explication of these alternative versions of the cone theorems. A few remarks concerning these issues are included in §6.

The use of degree theory in the following examples of solitary-wave problems automatically provides implications concerning the continuous dependence of solutions on parameters. This aspect can conveniently be summarized now in general terms, rather than being spelled out in the context of each problem. Accordingly, letting I denote some given interval of the real line, suppose $\{A_\gamma\}_{\gamma\in I}$ to be a one-parameter family of operators that are positive on a cone K and depend continuously on the real number γ in the sense that $d(A_\gamma x, A_\lambda x) \to 0$ as $\gamma \to \lambda \in I$, uniformly for $x \in K_r$ with some $r < 1$. Suppose also that, for each $\gamma \in I$, the operator A_γ satisfies the hypotheses of one or other of the preceding fixed-point theorems. If U is an open set in K that has no fixed point of A_λ on its boundary, then the triple (K, A_λ, U) is admissible in the sense of §2.2. From the general properties of the fixed-point index, it follows that (K, A_γ, U) is also admissible when γ is near λ and that then

$$i(K, A_\gamma, U) = i(K, A_\lambda, U).$$

This observation implies that, if $x_0 \in K$ is a fixed point of A_λ having a non-zero index, then A_λ has a fixed point $x \in K$ near x_0 provided γ is near λ. Furthermore, there is a neighbourhood U of x_0 in K and an interval $J \subset I$ containing λ such that the set

$$\Gamma = \{x \in U : \exists\, \gamma \in J \text{ with } A_\gamma x = x\}$$

is connected. If the fixed points of A_γ in K are locally unique, then the set Γ forms a continuous curve. To infer these properties, the additional information needed (e.g. the continuous dependence of the positive operator A on a parameter of the respective formulation) is immediately forthcoming in each of the examples that follow.

3. Solitary waves in one-dimensional nonlinear dispersive systems

Our first application relates to a class of model equations for waves propagating unidirectionally, while suffering nonlinear and dispersive effects. These equations have the general form

$$u_t + u_x + uu_x + Lu_t = 0, \tag{3.1}$$

in which the subscripts denote partial derivatives with respect to time t and the scalar position coordinate x, and L is a linear pseudo-differential operator with respect to x. This operator is symmetric and translation-invariant, and it transforms any constant into the zero function on $\mathbb{R}$. Thus, using the notation $\hat{v}(s) = \mathscr{F} v(x)$ for Fourier transforms, we have

$$(\widehat{Lv})\,(s) = \gamma(s)\,\hat{v}(s),$$

where the symbol $\gamma(s)$ of L is a real function satisfying $\gamma(-s) = \gamma(s)$ for all $s \in \mathbb{R}$ and $\gamma(0) = 0$. The initial-value problem for (3.1) and the physical origins of such equations were discussed by Benjamin *et al.* (1972), also by Benjamin (1974) and Bona (1980*a*, *b*, 1981*a*). Note that the particular specification $\gamma(s) = s^2$, and so $L = -\partial_x^2$ in (3.1), recovers the regularized long-wave equation mainly studied in the first-mentioned reference; and then replacement of ∂_t by $-\partial_x$ in the final term on the left-hand side, which is permissible within any scheme of approximation whereby (3.1) is obtained from physical examples, recovers the Korteweg–de Vries equation.

As mentioned in the introduction, the issue of existence of solitary-wave solutions of the model equation (3.1), or its more general analogue in which the nonlinearity appears in a form other than quadratic, has been addressed before. Precursors of the present, positive-operator technique were studied by Bona & Bose (1974) and Bona *et al.* (1976). A variational approach was followed by Benjamin (1974) to show existence of periodic travelling waves, and solitary waves were recovered from his theory in the limit as the period-length tends to infinity by Bona (1981*a*). Recently, Weinstein (1987) has used a variational approach and the method of concentrated compactness to establish an existence theory for solitary-wave solutions of (3.1) and of the Korteweg–de Vries-type version of (3.1) (cf. Bona 1980*a* and Albert *et al.* 1987 for a description of the setting for Weinstein's theory). It deserves remarking that there is no general uniqueness theory so far for these special solitary waves. We should acknowledge, however, that uniqueness has been proved by Amick & Toland (1990) in the special case of the Benjamin–Ono equation (Benjamin 1967). The stability theory for solitary-wave solutions of long-wave models is better developed than the uniqueness theory (cf. Benjamin 1972; Bona 1975; Scharf & Wreszinski 1981; Bennett *et al.* 1983; Shatah & Strauss 1985; Weinstein 1986, 1987; Grillakis *et al.* 1987; Bona *et al.* 1987; Albert *et al.* 1987).

We proceed formally to derive an integral equation satisfied by a solitary-wave solution of (3.1). Introduce $u = 2(c-1)\,\phi(x-ct)$ and presume the wave speed c to satisfy $c > 1$. As ∂_x and L correspond to multiplication of Fourier transforms by respective fixed symbols (functions of s alone), these operators formally commute. Hence an integration and use of the condition $\phi(x-ct) \to 0$ as $x \to \pm\infty$ leads at once to

$$\phi + c(c-1)^{-1}L\phi = \phi^2. \tag{3.2}$$

Henceforth the dependence of the solution on t can be left implicit, and we write $\phi = \phi(x)$ for simplicity. Assuming that the operator $I + c(c-1)^{-1}L$ can be inverted, we arrive at an integral equation for $\phi(x)$ expressed equivalently by

$$\begin{aligned}\phi &= [I + c(c-1)^{-1}L]^{-1}\phi^2 = B\phi^2\\ &= \int_{\mathbb{R}} k(x-y)\,\phi^2(y)\,\mathrm{d}y = A\phi, \quad \text{say.}\end{aligned} \tag{3.3}$$

The symbol of the linear operator denoted by B is evidently the even function

$$[1 + c(c-1)^{-1}\gamma(s)]^{-1} = \hat{k}(s) = \mathscr{F}k(x).$$

Our assumptions about the kernel $k(x)$ and its Fourier transform $\hat{k}(s)$ are as follows.

1. The function $\hat{k}(s)$ is positive, even, monotone decreasing on $(0, \infty)$ (i.e. $\gamma(s)$ is monotone increasing) and $\hat{k}(s) = O(|s|^{-\beta})$ with $\beta > 1$ as $|s| \to \infty$. Thus $\hat{k} \in L^1(\mathbb{R})$, from which it follows that $k = \mathscr{F}^{-1}\hat{k}$ is real-valued, even, and by the Riemann–Lebesgue lemma it is a bounded continuous function vanishing as $|x| \to \infty$.

2. The kernel $k(x)$ is positive on $\mathbb{R}$ and $k \in L^1(\mathbb{R})$. This assumption and (1) imply that the relation $\hat{k} = \mathscr{F}k$ holds pointwise, whence it follows that

$$\int_{\mathbb{R}} k(x)\,\mathrm{d}x = 1. \tag{3.4}$$

3. $k(x)$ is monotone decreasing on $(0, \infty)$, having moreover the property that there is a number $\lambda \geqslant 0$ such that $k(x)$ is strictly convex for $x \geqslant \lambda$.

Considered as a condition on $\gamma(s)$, (1) is plainly satisfied over a wide class of possible examples. Although necessary and sufficient conditions on $\gamma(s)$ to provide (2) and (3) are not known, these assumptions appear to be borne out in many examples and they are accordingly adopted as useful hypotheses. Note that (3.4) implies both the constant functions $\phi \equiv 1$ and $\phi \equiv 0$ to be solutions of (3.3).

Our aim on the basis of these assumptions is to establish a solitary-wave solution of (3.3), namely a fixed point of the nonlinear operator A that is distinct from the two trivial solutions, is positive, symmetric about zero, decreasing on $[0, \infty)$ and tending to zero as $|x| \rightarrow \infty$. We shall then show that any such solution ϕ of (3.3) is infinitely smooth, and that consequently it defines a travelling-wave solution $u = 2(c-1)\,\phi(x-ct)$ of (3.1). Note that the wave speed $c > 1$ has been prescribed and incorporated into the definition of A and the normalization of the solution ϕ, so that in principle a class of solitary waves with parameter c is to be covered. As will be seen, the problem presents considerable complexities, particularly as regards the need to discriminate between the required solution and the non-zero trivial solution, and the difficulties to be overcome are instructive in being typical of solitary-wave problems.

3.1. *Choice of Fréchet space and cone*

Let $C(\mathbb{R})$ denote the class of continuous, real-valued functions defined on $\mathbb{R}$. Being treated as a Fréchet space, $C(\mathbb{R})$ is assigned the topology of uniform convergence on bounded intervals of $\mathbb{R}$, and the generating family of semi-norms is taken to be

$$p_j(w) = \max_{-jl \leqslant x \leqslant jl} |w(x)|, \quad j = 1, 2, \ldots . \tag{3.5}$$

Here l is any fixed positive number. For the open ball of metric radius $r < 1$ centred on the zero element of $C(\mathbb{R})$, we write as in (2.2)

$$\mathscr{B}_r \equiv \mathscr{B}_r(0) = \{u \in C(\mathbb{R}) : d(0, u) < r\},$$

and for its boundary

$$\partial\mathscr{B}_r \equiv \partial\mathscr{B}_r(0) = \{u \in C(\mathbb{R}) : d(0, u) = r\}.$$

where the metric d is given by the formula (2.1).

The cone to be used is

$$K = \{u \in C(\mathbb{R}) : u(-x) = u(x) \geqslant 0 \quad \text{for each} \quad x \in \mathbb{R}, \quad u(x) \quad \text{is non-increasing with} \quad x \geqslant 0\}. \tag{3.6}$$

Note that if $u \in K$ and $d(0, u) = r < 1$, then $p_j(u) = p_1(u) = u(0)$ for all j, and consequently

$$r = u(0)/(1+u(0)), \quad u(0) = r/(1-r). \tag{3.7}$$

Thus K is p_1-bounded. It is also evidently closed. We shall again use the convenient notation

$$K_r = K \cap \mathscr{B}_r.$$

The second trivial solution of (3.3), $\phi = 1$, is a fixed point of A in the cone K, and this fact poses the main difficulties of the existence theory for the solitary-wave solution. But there is apparently no useful direct way to escape the complication of an extraneous solution. For instance, the set of positive even functions that are strictly monotone decreasing on $(0, \infty)$ excludes the function $\phi \equiv 1$, but this set is not a closed cone and so is useless for present purposes. It deserves emphasis that whereas the fixed-point theorems stated in §2.4 could have been stated and proved without requiring the cone in question to be closed, our use of the theorems depends crucially on this requirement, which appears to be generally unavoidable in applications. Specifically, K needs to be a closed cone so that the essential condition of compactness can be verified for the operator A, namely the condition that $A(K_r)$ has compact closure in K.

3.2. *Properties of the operator A*

To prepare for an application of theorem 2.8, the properties needing to be established are covered by two lemmas as follows.

LEMMA 3.1. *The operator A introduced in* (3.3) *maps K into K. The mapping is continuous; and for any r* $(0 < r < 1)$, $A(K_r)$ *is a relatively compact subset of K.*

Proof. To verify that $A(K) \subset K$, we take advantage of the evident fact that $u^2 \in K$ if $u \in K$. Thus it is only required to show that $B(K) \subset K$.

Because by assumption (2) the convolution kernel $k \in L^1(\mathbb{R})$ of the linear operator B is a positive, even and bounded function, so too is Bv if v is any non-negative, even and bounded function. Next, to show that Bv is a continuous on $\mathbb{R}$ if $v \in K$, we note from the definition

$$Bv(x) = \int_{\mathbb{R}} k(x-y)\, v(y)\, \mathrm{d}y \tag{3.8}$$

that, as $0 \leqslant v(y) \leqslant v(0) < \infty$ for each $y \in \mathbb{R}$, therefore

$$|Bv(x+h) - Bv(x)| \leqslant v(0) \int_{\mathbb{R}} |k(y-h) - k(y)|\, \mathrm{d}y.$$

As $k \in L^1(\mathbb{R})$ by assumption (2), it follows that

$$|Bv(x+h) - Bv(x)| \to 0 \quad \text{as} \quad h \to 0 \tag{3.9}$$

uniformly on $\mathbb{R}$, confirming Bv to be continuous.

Also directly from (3.8), by obvious changes of the variable of integration, we find that

$$Bv(x) - Bv(x+h) = \int_0^\infty \{k(y-x-\tfrac{1}{2}h) - k(y+x+\tfrac{1}{2}h)\}\{v(y-\tfrac{1}{2}h) - v(y+\tfrac{1}{2}h)\}\, \mathrm{d}y$$

whenever v as well as k is an even function. If $x \geqslant 0$ and $h > 0$, the first factor in the integrand is positive on $[0, \infty)$ by assumptions (2) and (3); and the second factor is non-negative on $[0, \infty)$ if $v \in K$ and $h > 0$. This plainly confirms Bv to be non-increasing with $x \geqslant 0$ if $v \in K$, and thus the demonstration that $Bv \in K$ if $v \in K$ is complete.

We next show A to be continuous from K into K. In what follows, suppose j to be a given positive integer, ϵ to be a given, arbitrarily small positive number and $\{u_n\}_{n=1}^{\infty}$ to be a sequence in K converging to u.

From the definition (3.3) of A, we have

$$\begin{aligned} |Au_n(x) - Au(x)| &\leqslant \int_{\mathbb{R}} k(x-y)\, |u_n(y) - u(y)|\, |u_n(y) + u(y)|\, \mathrm{d}y \\ &\leqslant C \int_{-\lambda}^{\lambda} k(x-y)\, |u_n(y) - u(y)|\, \mathrm{d}y + C^2 \int_{|y|>\lambda} k(x-y)\, \mathrm{d}y, \end{aligned} \tag{3.10}$$

where C is a number such that

$$\max_{y\in\mathbb{R},\,\pm} |u_n(y)\pm u(y)| \leqslant C \quad \text{for each} \quad n = 1,2,\ldots, \tag{3.11}$$

and $\lambda > 0$ is a number that is yet disposable. A finite number C complying with (3.11) clearly exists, because $Y = \{u\} \cup \{u_n : n = 1,2,\ldots\}$ is a compact and so bounded subset of K. For instance, if $Y \subset K_\rho$ where $0 < \rho < 1$, then according to (3.7) we can take $C = 2\rho/(1-\rho)$. With regard to the second term on the right-hand side of (3.10), the assumptions (1)–(3) about the kernel k imply that a number $\lambda_0 = \lambda_0(\epsilon, j)$ exists with the property

$$\max_{-jl\leqslant x\leqslant jl} \int_{|y|>\lambda_0} k(x-y)\,\mathrm{d}y < \frac{\epsilon}{2C^2}. \tag{3.12}$$

Accordingly, take $\lambda = \lambda_0$ and let i be the least integer satisfying $il \geqslant \lambda_0$. Because $\{u_n\}_{n=1}^\infty$ converges to u, we can choose N such that

$$p_i(u_n - u) < \epsilon/2C \quad \text{if} \quad n \geqslant N. \tag{3.13}$$

The inequalities (3.10), (3.12) and (3.13) together imply that

$$p_j(Au_n - Au) = \max_{-jl\leqslant x\leqslant jl} |Au_n(x) - Au(x)| < \tfrac{1}{2}\epsilon \int_{\mathbb{R}} k(x-y)\,\mathrm{d}y + \tfrac{1}{2}\epsilon = \epsilon.$$

Because by hypothesis ϵ is arbitrarily small, it follows from this result that $p_j(Au_n - Au) \to 0$ as $n \to \infty$, confirming that $A: K \to K$ is continuous.

Finally, regarding the relative compactness of $A(K_r)$, we consider an infinite sequence $\{u_n\}_{n=1}^\infty$ from K_r and need to verify that there is a subsequence of $\{Au_n\}_{n=1}^\infty$ converging in the metric d. By virtue of (3.4) and (3.7), the sequence of functions $\{Au_n\}_{n=1}^\infty$ has the uniform upper bound $[r/(1-r)]^2$ on $\mathbb{R}$, and the result (3.9) implies that these functions are equicontinuous. Accordingly, the Arzelà–Ascoli theorem and a Cantor diagonalization process completes the proof. □

As required for the application of theorem 2.8, the next lemma confirms that A expands the cone K.

Lemma 3.2. *Let* $0 < r < \tfrac{1}{2}$ *and* $\tfrac{1}{2} < R < 1$. *Then*

(*a*) $u \neq tAu$ *for each* $u \in K \cap \partial B_r$ *and* $t \in [0,1]$, *and, if* R *is sufficiently close to* 1,

(*b*) $u - Au \neq a\mathbf{1}$ *for each* $u \in K \cap \partial \mathscr{B}_R$ *and* $a \geqslant 0$, *where* $\mathbf{1}$ *denotes the constant function that takes the value* 1 *for every* $x \in \mathbb{R}$.

Proof. (*a*) Suppose to the contrary that an element $u \in K$, $d(0,u) = r$, and a number $t \in [0,1]$ exist such that $u = tAu$. It follows that

$$0 < u(0) = \int_{\mathbb{R}} k(y)\,u^2(y)\,\mathrm{d}y \leqslant u^2(0),$$

and therefore $u(0) \geqslant 1$. But according to (3.7) this contradicts the condition $r < \tfrac{1}{2}$, and so the proof of (*a*) is complete.

(*b*) Supposing the contrary, we have an element $u \in K$, $d(0,u) = R$, and an $a \geqslant 0$ such that $a\mathbf{1} + Au = u$. Therefore

$$a + \int_0^1 Au\,\mathrm{d}x = \int_0^1 u\,\mathrm{d}x \leqslant \left[\int_0^1 u^2\,\mathrm{d}x\right]^{\frac{1}{2}}. \tag{3.14}$$

But the facts that k is a positive and symmetric decreasing function and that u is even imply

$$\int_0^1 Au\,dx = \int_0^1\left[\int_{\mathbb{R}} k(x-y)\,u^2(y)\,dy\right]dx \geqslant \int_0^1\left[\int_0^1 [k(x-y)+k(x+y)]\,dy\right]u^2(y)\,dy \geqslant \alpha S^2, \tag{3.15}$$

where
$$\alpha = \int_0^2 k(z)\,dz \quad \text{and} \quad S^2 = \int_0^1 u^2(y)\,dy.$$

The combination of (3.14) and (3.15) gives

$$\alpha S^2 - S + a \leqslant 0,$$

showing that
$$S \leqslant \alpha^{-1} \quad \text{and} \quad a \leqslant \tfrac{1}{4}\alpha^{-1}. \tag{3.16}$$

Now, because u is positive, even and non-increasing on $[0, \infty)$, it is clear that

$$Au(x) \equiv \sum_{m=-\infty}^{\infty} \int_{-1}^{1} k(x-y-2m)\,u^2(y+2m)\,dy \leqslant \int_{-1}^{1} \tilde{k}(x-y)\cdot u^2(y)\,dy, \tag{3.17}$$

where $\tilde{k}$ is the positive periodic function with period 2 defined by

$$\tilde{k}(x) = \sum_{m=-\infty}^{\infty} k(x-2m).$$

Because $k \in C_b(\mathbb{R}) \cap L^1(\mathbb{R})$ and k is even and non-increasing on $[0, \infty)$, $\tilde{k}$ is well defined as an even, continuous and bounded function. Hence, writing

$$\tilde{k}_m = \max_{-1\leqslant x \leqslant 1} \tilde{k}(x) = \sum_{m=-\infty}^{\infty} k(2m),$$

we have from (3.17) and then the first of (3.16) that

$$Au(x) \leqslant 2\tilde{k}_m S^2 \leqslant 2\tilde{k}_m \alpha^{-2}.$$

This result and the second of (3.16) imply that, for each $x \in \mathbb{R}$,

$$u(x) = a + Au(x) \leqslant \tfrac{1}{4}\alpha^{-1} + 2\tilde{k}_m \alpha^{-2} = c, \quad \text{say.}$$

On the other hand, the property (3.7) means here that $u(0) = R/(1-R)$. If $R > c/(1+c)$, a contradiction is thus established, which completes the proof of (b). □

3.3. *Existence theorem*

Since the conditions (2.13) and (2.15) of theorem 2.8 have been verified, it follows that A has a fixed point in the conical segment $K_r^R = \{u \in K : r < d(0, u) < R\}$, and the index of A on K_r^R is -1. Our analysis is far from complete, however, because the trivial fixed point $\phi = \mathbf{1}$ lies in K_r^R (as $d(0, 1) = \frac{1}{2}$). We shall proceed to show that if there were only one fixed point in K_r^R, then its index would necessarily be zero, and thus a contradiction is demonstrated proving the existence of a third, non-trivial fixed point of A in K. This part of the theory is the most difficult, proceeding by a series of lemmas to the main theorem.

The core of the argument will consist in showing that, on a small metric ball centred at $\mathbf{1} \in K$, the operator A is homotopic to an operator whose non-trivial fixed points represent periodic waves, with a prescribed period that is sufficiently large. The index of $\mathbf{1}$, considered as a fixed point of the periodic-wave operator, is then shown to be zero. For the periodic-wave problem to which reference has thus to be made, it is natural to consider the cone

$$K_l = \{u \in C(\mathbb{R}) : u(-x) = u(x) \geqslant 0 \ \forall x \in \mathbb{R},\ u \text{ is } 2l\text{-periodic and non-increasing on } [0,l]\},$$

which like K is a closed normal cone in $C(\mathbb{R})$. In the present application, the half-period l is identified with the number l introduced in the definition (3.5) and (2.1) of the metric d.

The operator $r_l : K \to K_l$ defined by

$$(r_l u)(x) = \begin{cases} u(x) & \text{if } 0 \leqslant |x| \leqslant l, \\ u(2ml - |x|) & \text{if } (2m-1)\,l \leqslant |x| \leqslant (2m+1)\,l, \quad m = 1, 2, \ldots, \end{cases}$$

is easily seen to be continuous. In fact, $r_l u$ is just the periodic extension of u that has period $2l$ and equals u on $[-l, l]$. Using the terminology introduced in §2, we can assert that the cone K r_l-dominates K_l, because the operator $s_l : K_l \to K$ defined by

$$(s_l v)(x) = \begin{cases} v(x) & \text{if } 0 \leqslant |x| \leqslant l, \\ v(l) & \text{if } |x| \geqslant l \end{cases}$$

is also continuous and plainly $r_l \circ s_l$ is the same as the identity map on K_l.

In view of assumptions (1) and (2) about the kernel k of the linear operator B, the defintion (3.3) evidently covers an interpretation of B and A as operators on functions $v \in K_l$ and other functions $v \in C(\mathbb{R})$ that are even and periodic with period $2l$. From (3.3) and its context, alternative representations of B as an operator on functions in this class are seen to be

$$Bv(x) = \int_0^\infty \{k(x-y) + k(x+y)\}\, v(y)\, \mathrm{d}y = \int_0^l k_l(x,y)\, v(y)\, \mathrm{d}y, \tag{3.18}$$

where

$$k_l(x,y) = \sum_{m=-\infty}^{\infty} \{k(x-y-2ml) + k(x+y+2ml)\}$$
$$= \frac{1}{l}\left\{1 + 2\sum_{n=1}^{\infty} \hat{k}(n\pi/l)\cos(n\pi x/l)\cos(n\pi y/l)\right\}. \tag{3.19}$$

We now establish various facts that will be required relating to the periodic problem.

LEMMA 3.3. *The operator A constitutes a continuous mapping of K_l into itself. If $0 < r < 1$, then $A(K_l \cap \mathscr{B}_r)$ is a relatively compact subset of K_l.*

Proof. Based either on the original definition of B and A in (3.3) or on the alternative representation (3.18) of B, confirmation that A is a continuous and compact mapping of K_l into itself follows straightforwardly on the same lines as the proof of lemma 3.1 where A was considered as a mapping of K. □

The next of the needed facts is obvious, being presented as a lemma because it is used repeatedly in what follows.

LEMMA 3.4. *Let u be a continuous real-valued function that is non-increasing on $[0, l]$, not necessarily being positive or differentiable. Then*

$$\int_0^l u(x)\cos(\pi x/l)\,\mathrm{d}x \geqslant 0. \tag{3.20}$$

Moreover, equality holds in (3.20) only if $u(x) = $ const. for $0 \leqslant x \leqslant l$.

Proof. The inequality follows immediately upon recognition that the integral is the same as the integral of $\{u(x)-u(l-x)\}\cos(\pi x/l)$ from 0 to $\frac{1}{2}l$, in which range $\cos(\pi x/l) \geqslant 0$ and $u(x) \geqslant u(\frac{1}{2}l) \geqslant u(l-x)$. Unless $u(x) = \text{const.}$ on $[0,l]$, we have $u(x)-u(l-x) > 0$ on a set of positive measure in $[0,\frac{1}{2}l)$ where $\cos(\pi x/l) > 0$. □

Another needed result depends on the assumptions (1)–(3) about the kernel of the linear operator B.

LEMMA 3.5. *If l is chosen sufficiently large and $y > l$, then*

$$\int_0^{\frac{1}{2}l} \{k(y+x-l)-k(y-x)-k(y+x)+k(y-x+l)\}\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x > 0.$$

Proof. Let $\sigma = \sigma(x,y,l)$ denote the expression in curly brackets in the above integral. Suppose in the first place that $l > 2\lambda$, where λ is the number introduced in the definition of property (3). Accordingly, consider

$$\int_0^{\frac{1}{2}l} \sigma(x,y,l)\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x = \int_0^{\lambda} \sigma(x,y,l)\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x + \int_{\lambda}^{\frac{1}{2}l} \sigma(x,y,l)\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x,$$

and note that, if $\lambda < x < \frac{1}{2}l$, then

$$y-x+l > y+x > y-x > y+x-l > \lambda. \tag{3.21}$$

Moreover, the first two and the last two of these four arguments of k appearing in the factor $\sigma(x,y,l)$ of the integrand are equally spaced. Hence the positivity of $k(x)$ and its strict convexity for $x \geqslant \lambda$ implies that $\sigma(x,y,l) > 0$ in the second integral, which is therefore positive.

Regarding the first integral, we must distinguish the two cases (i) $y > l+\lambda$ and (ii) $l < y \leqslant l+\lambda$. In case (i) the inequalities (3.21) still all hold for $0 < x \leqslant \lambda$, and so the first integral too is positive by the same reasoning as before. In case (ii), whatever the choice of $l > 2\lambda$, we have that

$$0 < y+x-l \leqslant 2\lambda,$$

whence

$$k(y+x-l) \leqslant k(2\lambda) \tag{3.22}$$

by the monotonicity of k; and we also have that

$$k(y-x+l) \geqslant 0, \quad k(y+x) \leqslant k(y-x) < k(l-\lambda). \tag{3.23}$$

Because $k \in L^1(\mathbb{R})$, it is plain that l can be chosen large enough to make

$$k(l-\lambda) < \tfrac{1}{2}k(2\lambda).$$

Hence, in view of (3.22) and (3.23), this revised choice of l ensures that the first integral is positive in case (ii) also, and thus the proof is complete. □

We are now in a position to prove the anticipated result concerning the index of the second trivial solution.

LEMMA 3.6. *If* $\mathbf{1}$ *were the only fixed point of A in K_r^R with $0 < r < \frac{1}{2} < R < 1$, then the index of this fixed point would be zero, i.e.* $i(A,\mathbf{1}) = 0$.

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Proof. We consider the homotopy

$$H(u,t) = tAu + (1-t)\, s_l A r_l u$$

on
$$K \cap \mathscr{B}_\epsilon(\mathbf{1}) = r_l^{-1}[K_l \cap \mathscr{B}_\epsilon(\mathbf{1})], \tag{3.24}$$

where ϵ is a sufficiently small positive number. In particular, ϵ is chosen small enough for the closure of the set (3.24) to lie in K_r^R. Also, the number l is chosen large enough for the result in lemma 3.5 to apply and for $\hat{k}(\pi/l) > \frac{1}{2}$. Because $H: K \times [0,1] \to K$ is compact by virtue of the compactness of A when considered as a mapping of K (lemma 3.1) and when considered as a mapping of K_l (lemma 3.3), it will follow that A is homotopic to $s_l A r_l$ on the set (3.24) if there is no element of

$$\partial[K \cap \mathscr{B}_\epsilon(\mathbf{1})] = \{u \in K : d(\mathbf{1}, u) = \epsilon\} \tag{3.25}$$

such that
$$u = tAu + (1-t)\, s_l A r_l u \tag{3.26}$$

for any choice of $t \in [0,1]$. On the supposition that such an element exists satisfiying (3.26), we infer at once that $t < 1$ because A is presumed to have no fixed point in the set (3.24). The case $t = 0$, implying $s_l A r_l$ to have a fixed point in $\partial[K_l \cap \mathscr{B}_\epsilon(\mathbf{1})]$, has yet to be excluded.

For $0 \leqslant x \leqslant l$, equation (3.26) reduces to

$$u(x) = t(Au)(x) + (1-t)(Ar_l u)(x) = B[(r_l u)^2](x) - t\psi(x), \tag{3.27}$$

where
$$\psi(x) = (B\{(r_l u)^2 - u^2\})(x) = \int_{\mathbb{R}} k(x-y)\{(r_l u)^2(y) - u^2(y)\}\,dy$$
$$= \int_{y>l} \{k(x-y) + k(x+y)\}\{(r_l u)^2(y) - u^2(y)\}\,dy.$$

Multiplying each of the terms in (3.27) by $\cos(\pi x/l)$ and integrating from 0 to l, we obtain from the first term on the right-hand side

$$\int_0^l \cos\left(\frac{\pi x}{l}\right) B[(r_l u)^2]\,(x)\,dx = \int_0^l u^2(x)\, B\left\{\cos\left(\frac{\pi x}{l}\right)\right\} dx$$
$$= \hat{k}\left(\frac{\pi}{l}\right) \int_0^l u^2(x) \cos\left(\frac{\pi x}{l}\right) dx.$$

(These equalities are clarified by the representation (3.18) of B, with kernel expressed by (3.19).) Hence (3.27) shows that

$$\int_0^l \left\{\hat{k}\left(\frac{\pi}{l}\right) u^2(x) - u(x)\right\} \cos\left(\frac{\pi x}{l}\right) dx = t\int_0^l \psi(x) \cos\left(\frac{\pi x}{l}\right) dx. \tag{3.28}$$

Now, $u(x)$ is non-increasing with x in $[0,l]$, and consequently so is $\hat{k}(\pi/l)\,u^2(x) - u(x)$ in the first integrand if $2\hat{k}(\pi/l)\,u(l) \geqslant 1$, which condition is plainly satisfied if

$$p_1(\mathbf{1} - u) = \max_{0 \leqslant x \leqslant l} |1 - u(x)| \leqslant \frac{2\hat{k}(\pi/l) - 1}{2\hat{k}(\pi/l)}.$$

Because
$$\epsilon = d(1,u) \geqslant \frac{p_1(1-u)}{2\{1 + p_1(1-u)\}},$$

it follows that $\hat{k}(\pi/l)\,u^2-u$ is a non-increasng function on $[0,l]$ if a positive value of $\epsilon<\frac{1}{2}$ is chosen to satisfy
$$2\epsilon/(1-2\epsilon)\leqslant\{2\hat{k}(\pi/l)-1\}/2\hat{k}(\pi/l),$$
i.e.
$$\epsilon\leqslant\{2\hat{k}(\pi/l)-1\}/\{8\hat{k}(\pi/l)-2\}. \tag{3.29}$$
According to lemma 3.4, the integral on the left-hand side of (3.28) is then non-negative, being moreover positive unless $u=$ const. on $[0,l]$.

In contrast, after substitution of the integral expressing $\psi(x)$, the right-hand side of (3.28) is seen to equal
$$t\int_{y>l}\zeta(y)\{(r_l u)^2(y)-u^2(y)\}\,\mathrm{d}y, \tag{3.30}$$
where
$$\begin{aligned}\zeta(y)&=\int_0^l\{k(x-y)+k(x+y)\}\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x\\&=-\int_0^{\frac{1}{2}l}\{k(l-x-y)+k(l-x+y)-k(x-y)-k(x+y)\}\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x\end{aligned}$$
is negative according to lemma 3.5 for $y>l$. The factor in curly brackets in the integrand of (3.30) is positive for almost all $y>l$ unless
$$u(y)=\text{const.}\quad\text{for all } y\in\mathbb{R},$$
and this exceptional case is impossible because (3.26) would then require the constant, say α, to satisfy $\alpha=\alpha^2$, which excludes the constant elements of the set (3.25) if $0<\epsilon<\frac{1}{2}$. If $t>0$, it follows that the right-hand side of (3.28) is negative, and a contradiction is thus established. If $t=0$, equation (3.28) is still contradictory according to lemma 3.4 unless $u(x)=$ const. for $0\leqslant x\leqslant l$, which case has already been excluded. These contradictions complete the demonstration that A is homotopic to $s_l A r_l$ on the set (3.24).

The fact that K r_l-dominates K_l hence implies that
$$i(K,A,r_l^{-1}[K_l\cap\mathscr{B}_\epsilon(\mathbf{1})])=i(K_l,A,K_l\cap\mathscr{B}_\epsilon(\mathbf{1})), \tag{3.31}$$
and accordingly we can complete the proof by showing the second index to be zero. Let $u^*\in K_l$ be chosen such that it is strictly decreasing on $(0,l)$. By lemma 2.7 the present aim will be accomplished by verifying that for all $u\in K_l$ with $d(\mathbf{1},u)=\epsilon>0$ sufficiently small, and for all $a\geqslant0$, the relation
$$u-Au=au^* \tag{3.32}$$
is impossible.

Supposing that there is such a solution of (3.32), we proceed as before in the treatment of (3.27) and obtain from (3.32)
$$-\int_0^l\left\{\hat{k}\left(\frac{\pi}{l}\right)u^2(x)-u(x)\right\}\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x=\int_0^l au^*(x)\cos\left(\frac{\pi x}{l}\right)\mathrm{d}x,$$
the right-hand side of which is strictly positive by lemma 3.4 if $a>0$. In contradiction, the same lemma shows the left-hand side to be non-positive if ϵ satisfies (3.29), which condition ensures that $\hat{k}(\pi/l)\,u^2-u$ like u is a non-increasing function on $(0,l)$. The case $a=0$ is also contradictory according to lemma 3.4 for the reason explained with reference to (3.28) in the

case $t = 0$. These contradictions establish that $i(K_l, A, K_l \cap \mathscr{B}_\epsilon(\mathbf{1})) = 0$. Hence in view of the identity (3.24) and the proven equality (3.31), we have that

$$i(K, A, r_l^{-1}[K_l \cap \mathscr{B}_\epsilon(\mathbf{1})]) \equiv i(A, \mathbf{1}) = 0,$$

and the proof of lemma 3.6 is complete. □

The final result can now be given.

THEOREM 3.7. *There exists a* $\phi \in K_r^R$ *such that* $\phi = A\phi$ *and*

$$\lim_{|x|\to\infty} \phi(x) = 0. \tag{3.33}$$

Proof. Having been shown by lemmas 3.1 and 3.2 to be applicable, theorem 2.8 implies that if the constant function $\mathbf{1}$ were the only fixed point of A in K_r^R, then $i(A, \mathbf{1}) = -1$. But lemma 3.6 has established that $i(A, \mathbf{1}) = 0$, and thus a fixed point ϕ is guaranteed that is not a constant.

To confirm the property (3.33), we first note that the attribution $\phi \in K$ ensures that

$$\lim_{|x|\to\infty} \phi(x) = c,$$

where c is a non-negative constant. The value of c is now seen to be necessarily either 0 or 1. To this end, note that

$$\phi^2(x+n) - c^2 \to 0$$

for each x as $n \to \infty$ and therefore, by the dominated-convergence theorem, we also have

$$\begin{aligned} 0 = \lim_{n\to\infty} \int_{\mathbb{R}} k(x)\{\phi^2(x+n) - c^2\}\,\mathrm{d}x &= \lim_{n\to\infty} \int_{\mathbb{R}} k(x-n)\,\phi^2(x)\,\mathrm{d}x - c^2 \int_{\mathbb{R}} k(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \phi(n) - c^2 = c - c^2, \end{aligned}$$

implying that $c = c^2$. It remains to show that $c \neq 1$.

Supposing $c = 1$, we would have $v = \phi - \mathbf{1} \in K$, and from (3.3) the equation satisfied by v is seen to be

$$v(x) = \int_{\mathbb{R}} k(x-y)\{2v(y) + v^2(y)\}\,\mathrm{d}y. \tag{3.34}$$

Now, (3.4) and the other properties of k plainly imply that there is a finite positive number m such that

$$\int_{-m}^{m} k(x)\,\mathrm{d}x > \tfrac{3}{4},$$

and accordingly
$$\int_{-4m}^{4m} k(x-y)\,\mathrm{d}x > \tfrac{3}{4} \quad \text{if} \quad -3m \leqslant y \leqslant 3m.$$

Because $v \in K$ $(v \not\equiv 0)$, it follows that

$$\int_{-4m}^{4m} \left(\int_{\mathbb{R}} k(x-y)\,v(y)\,\mathrm{d}y \right) \mathrm{d}x \geqslant \int_{-4m}^{4m} \int_{-3m}^{3m} k(x-y)\,v(y)\,\mathrm{d}y\,\mathrm{d}x > \frac{3}{4} \int_{-3m}^{3m} v(y)\,\mathrm{d}y > \frac{9}{16} \int_{-4m}^{4m} v(y)\,\mathrm{d}y.$$

(The last inequality follows because the values of $v(y)$ in $[-4m, -3m]$ and $[3m, 4m]$ do not exceed any in $[-3m, 3m]$.) This result is in contradiction of (3.34), because the equation requires the last integral, without the factor $\frac{9}{16} > \frac{1}{2}$, to be greater than twice the first integral. Thus the proof of the existence theorem is complete. □

3.4. *Regularity*

It remains to show that a solitary-wave solution of the integral equation is in fact an infinitely differentiable function, all of whose derivatives vanish in the limit $|x| \to \infty$. The assumed properties of the operator A, particularly those of the linear operator B in its composition, are insufficient for any immediate conclusion about differentiability to be available from the basic attribution $\phi \in C(\mathbb{R})$ and the fact that $\phi = A\phi$. Resort has to be made instead to arguments in terms of Sobolev spaces, and for this purpose the following result is pivotal.

Lemma 3.8. *Let $\phi(x)$ be any solution of* (3.3) *in K satisfying* (3.33). *Then $\phi \in L^p(\mathbb{R})$ for all $p \geqslant 1$.*

Proof. We shall first show that $x\phi(x)$ is bounded on $[0, \infty)$. As $\phi \in K \subset C(\mathbb{R})$, this property at once ensures that $\phi \in L^p(\mathbb{R})$ for $p > 1$. Then, as $\phi \in L^2(\mathbb{R})$ and consequently $B\phi^2 \in L^1(\mathbb{R})$ according to assumption (2) about the kernel k of the linear operator B, it also follows that $\phi \in L^1(\mathbb{R})$.

For any $m > 0$, we have

$$\int_0^m \phi(x)\,\mathrm{d}x = \int_0^m \int_0^\infty \{k(x-y)+k(x+y)\}\,\phi^2(y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^\infty \alpha_m(y)\,\phi^2(y)\,\mathrm{d}y, \tag{3.35}$$

where

$$\alpha_m(y) = \int_0^m \{k(x-y)+k(x+y)\}\,\mathrm{d}x.$$

It is evident from the properties (1) and (2) of k that $\alpha_m(y) \uparrow 1$ as $m \to \infty$, uniformly on compact subsets of $[0, \infty)$. As $\phi \in K$ and because of its asymptotic property (3.33), there is a finite number $\mu > 0$ such that $\phi(x) \leqslant \frac{1}{2}$ if $x \geqslant \mu$. Accordingly, taking $m > \mu$ and rearranging (3.35), we obtain

$$I_m = \int_0^\mu \{\alpha_m(y)\,\phi^2(y) - \phi(y)\}\,\mathrm{d}y$$

$$= \int_\mu^m \{\phi(y) - \alpha_m(y)\,\phi^2(y)\}\,\mathrm{d}y - \int_m^\infty \alpha_m(y)\,\phi^2(y)\,\mathrm{d}y.$$

As μ is fixed, the aforesaid property of $\alpha_m(y)$ implies that the first integral, denoted by I_m, is bounded independently of m, in fact converging to its upper bound

$$I_\infty = \int_0^\mu \{\phi^2(y) - \phi(y)\}\,\mathrm{d}y$$

as $m \to \infty$.

On the other hand, we have

$$\int_m^\infty \alpha_m(y)\,\phi^2(y)\,\mathrm{d}y \leqslant \phi^2(m)\int_m^\infty \alpha_m(y)\,\mathrm{d}y$$

$$= \phi^2(m)\int_m^\infty\int_0^m \{k(x-y)+k(x+y)\}\,\mathrm{d}x\,\mathrm{d}y \leqslant m\phi^2(m)$$

in view of (3.4), and therefore

$$I_\infty \geqslant \int_\mu^m \{\phi(y)-\alpha_m(y)\,\phi^2(y)\}\,\mathrm{d}y - m\phi^2(m).$$

But $\alpha_m(y) \leqslant 1$ for all m and y, and $\phi(y) \leqslant \frac{1}{2}$ on $[\mu, \infty)$. Hence

$$I_\infty \geqslant \frac{1}{2}\int_\mu^m \phi(y)\,\mathrm{d}y - m\phi^2(m) \geqslant \tfrac{1}{2}(m-\mu)\,\phi(m) - m\phi^2(m) = m\phi(m)\,\{\tfrac{1}{2}-(\mu/2m)-\phi(m)\}. \quad (3.36)$$

For m large, the factor in brackets on the right-hand side of (3.36) is positive. A number m_0 can be found such that this factor is bounded below by $\frac{1}{4}$, say, if $m \geqslant m_0$. Thus it appears that

$$m\phi(m) \leqslant 4I_\infty \quad \text{if} \quad m \geqslant m_0,$$

and the needed bound is demonstrated. For the simple reasons explained at the start, the proof of the lemma is thereby complete. □

As was noted implicitly in the first paragraph of the proof, the attribution $\phi \in L^2(\mathbb{R})$ established by lemma 3.8 substantiates the identity

$$\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \phi^2(x)\,\mathrm{d}x$$

satisfied by the solitary-wave solution. It is noteworthy that this identity confirms a property strongly suggested by earlier stages of the analysis, particularly the proof of lemma 3.6. Namely, over some central interval $(-x_0, x_0)$, say, $\phi(x)$ has values above those of the second trivial solution **1**. The fact that the difference between the two non-zero solutions has both positive and negative values exemplifies a general property demonstrated in abstract by Krasnosel'skii (1964*a*, §6.3.2.), which is attributable to multiple non-zero fixed points of any 'convex' nonlinear operator that maps a cone into itself [i.e. according to Krasnosel'skii's definition, an operator like the present A, also the one to be considered in §4, that for some positive number $\eta = \eta(u,t)$ satisfies $t(1-\eta)\,Au - A(tu) \in K$ for all $u \in K$ and $t \in (0,1)$].

We can now present the main result concerning the regularity of ϕ.

THEOREM 3.9. *Any solitary-wave solution of* (3.3) *in* K *satisfying* (3.33) *has the attribution* $\phi \in H^\infty(\mathbb{R})$ (*i.e.* ϕ *is a function in* $C^\infty(\mathbb{R})$, *all of whose derivatives are like* ϕ *itself in* $L^2(\mathbb{R})$, *vanishing in the limit* $|x| \to \infty$). *Moreover,* ϕ *and all its derivatives lie in the domain* $D(L) = \{w \in L^2(\mathbb{R}) : Lw \in L^2(\mathbb{R})\}$ *of the operator* L *introduced in* (3.1), *so that the function* $u(x,t) = 2(c-1)\,\phi(x-ct)$ *is a smooth, travelling-wave solution of* (3.1).

Proof. Lemma 3.8 establishes that $\phi \in L^4(\mathbb{R})$, which means that $\psi = \phi^2 \in L^2(\mathbb{R})$. Therefore the Fourier transform $\hat{\psi}$ of ψ is also in $L^2(\mathbb{R})$. By virtue of the assumption (1) about the kernel of the linear operator B, it follows that

$$\phi = B\psi \in D(L) \subset H^\alpha(\mathbb{R}) \quad \text{for any} \quad \alpha \leqslant \beta.$$

(Here $\phi \in D(L)$ follows because, by Parseval's inequality,

$$\|L\phi\|_0^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\gamma(s)\,\hat{k}(s)\,\hat{\psi}(s)|^2\,\mathrm{d}s$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \left|\frac{\gamma(s)\,\hat{\psi}(s)}{1+c(c-1)^{-1}\gamma(s)}\right|^2 \mathrm{d}s \leqslant \left(\frac{c-1}{c}\right)^2 \|\psi\|_0^2,$$

where $\|\cdot\|_0$ denotes the $L^2(\mathbb{R})$ norm; and $D(L) \subset H^\alpha(\mathbb{R})$ for $\alpha \leqslant \beta$ follows because assumption (1) implies that, for large values of $|s|$, $\gamma(s) \geqslant \kappa|s|^\beta$ with κ a positive constant (cf. Adams 1975, §§5.23 and 7.57).)

Extending the argument inductively confirms both that $\phi \in H^\infty(\mathbb{R})$ and that $(\partial_x)^j\phi \in D(L)$ for $j = 0, 1, 2, \ldots$. The statement concerning $u = 2(c-1)\,\phi(x-ct)$ is hence verified by application of $I + c(c-1)^{-1}L$ to the equation $\phi = B\phi^2$, which gives

$$[I + c(c-1)^{-1}L]\,\phi = \phi^2.$$

Differentiating this identity and using the now plainly established fact that $L\,\partial_x\phi = \partial_x L\phi$, we have

$$[I + c(c-1)^{-1}L]\,\phi' = 2\phi\phi',$$

which shows that the specified function u satisfies (3.1) in the sense that each term of this equation is attributable as an element of $H^\infty(\mathbb{R})$. □

A simple corollary is that $\phi \in W^{\infty,1}(\mathbb{R})$ (i.e. ϕ and all its derivatives are in $L^1(\mathbb{R})$). It has been shown in the proof of lemma 3.8 that $\phi \in L^1(\mathbb{R})$ follows immediately from $\phi \in L^2(\mathbb{R})$, and we have

$$\phi' = 2B(\phi\phi'),$$

the right-hand side of which is in $L^1(\mathbb{R})$, because $\phi, \phi' \in L^2(\mathbb{R})$ and B is a convolution with an L^1 kernel. For corresponding reasons, we have

$$\phi'' = 2B(\phi\phi'' + \phi'^2) \in L^1(\mathbb{R}),$$

and the argument can be continued indefinitely using the chain rule.

4. Solitary waves in continuously stratified fluids

The next example arises from the analysis of steady wave motions in a heavy fluid that is inviscid, heterogeneous but incompressible and is confined between horizontal planes. The problem to be treated has been studied previously by other methods, and explanations of its hydrodynamic origins may be found elsewhere. For example, a comprehensive approximate theory describing periodic and solitary waves of small but finite amplitude in stably stratified layers of fluid was given by Benjamin (1966), and a rigorous basis for the small-amplitude theory was the objective of the paper by Ter-Krikorov (1963) using the contraction-mapping principle. Further mathematical specifications, such that extend the theory for the modelling of large-amplitude waves, were discussed by Benjamin (1971, §6) in his account of conjugate flows independent of the horizontal coordinate. Reference may be made to these previous papers for the physical meanings of the specifications adopted here.

Subsequent to the cited paper by Ter-Krikorov, which constituted the first precise attack on the class of semilinear problems represented in part by (4.1) and (4.2) below, the mathematical theory of internal waves has been extended by several contributions free from restriction to small amplitudes. The pioneering step in this direction was made by Turner (1981), and then Bona *et al.* (1983) developed a global theory for periodic and solitary waves in systems of the type to be reconsidered here. Amick (1984) and Amick & Toland (1983) have presented an exhaustive treatment of this problem. Most recently, Amick & Turner (1986, 1989) and Bona & Sachs (1989) have studied the related problem of internal solitary waves in two-layer systems.

Describing steady waves in an appropriate frame of reference, the normalized perturbation $\phi(x,y)$ of a pseudo-stream-function satisfies the semilinear elliptic equation

$$\Delta\phi + \phi f(y,\phi) = 0 \quad \text{in} \quad \mathbb{R}\times(0,1) \tag{4.1}$$

and the boundary conditions

$$\phi(x,0) = 0, \quad \phi(x,1) = 0. \tag{4.2}$$

The wave velocity and the prescribed density structure for the fluid at rest enter the specification of the function in f in (4.1), and it is determined that the trivial solution $\phi \equiv 0$ of (4.1) and (4.2) corresponds to a supercritical state of flow, characterized by the property that infinitesimal periodic waves cannot be superposed upon it (cf. condition (IV) below). Our present aim is to establish a solitary-wave solution of (4.1) and (4.2), namely a non-trivial solution ϕ with the properties (i) $\phi(x,y)$ is non-negative on $\mathbb{R}\times[0,1]$, (*ii*) $\phi(x,y)$ is even in x and non-increasing with $|x|$, and (iii) $\lim_{x\to\pm\infty}\phi(x,y) = 0$. Note from (4.1) and (4.2) that if $\phi(x,y)$ is a solution, so is $\phi(x+a)$ for any real number a. Accordingly, the prescribed property (ii) includes a convenient normalization.

The function f is assumed to satisfy the following conditions, which were explained by Benjamin (1971, §6) to specify a rational model for perturbations relative to a supercritical base-flow and were exemplified explicitly in his account. Other examples providing explicit forms of f were considered by Bona *et al.* (1983).

(I) The function $f(s,t):[0,1]\times[0,\infty)\to[0,\infty)$ is Hölder continuous in both variables, having moreover a continuous partial derivative with respect to the second variable t.

(II) For all $(s,t)\in[0,1]\times[0,\infty)$, we have $f+t(\partial f/\partial t) < 4\pi^2$.

(III) For each $s\in[0,1]$, $f(s,t)$ is strictly monotonic increasing with $t \geqslant 0$.

(IV) For the Sturm–Liouville problem

$$\left.\begin{aligned} \mathrm{d}^2\xi/\mathrm{d}y^2 + \lambda^{-1}f(y,0)\,\xi = 0,\\ \xi(0) = \xi(1) = 0, \end{aligned}\right\} \tag{4.3}$$

whose characteristic values λ are all real and positive (as is well known), the largest characteristic value λ_1 satisfies $\lambda_1 < 1$.

(V) For each $s\in[0,1]$, $f(s,t)$ increases to a function $Q(s)$ as $t\to\infty$. The function Q is continuous from $[0,1]$ to $(0,\infty)$ and such that the largest characteristic value μ_1 of the Sturm–Liouville problem

$$\left.\begin{aligned} \mathrm{d}^2\eta/\mathrm{d}y^2 + \mu^{-1}Q(y)\,\eta = 0,\\ \eta(0) = \eta(1) = 0 \end{aligned}\right\} \tag{4.4}$$

satisfies $\mu_1 > 1$.

It was proved by Benjamin (1971, §6) that these conditions guarantee the existence of a non-trivial x-independent solution of (4.1) and (4.2), say $\phi = \Phi(y)$, which is positive on

$0 < y < 1$ and in fact represents a subcritical uniform flow conjugate to the supercritical base-flow. As in the previous example, allowance for this extraneous solution will be a central issue of the solitary-wave theory. Conditions (II) and (III) were noted by Benjamin to imply, moreover, that Φ is unique in its category. As several steps in the present analysis will depend on this finding, confirmation of it is included as follows.

Suppose for the sake of contradiction that $\Phi_1(y)$ and $\Phi_2(y)$ are two distinct non-trivial solutions of (4.1) and (4.2) that are both non-negative and independent of x. Then (4.1) and (4.2) show that

$$\int_0^1 \Phi_1\,\Phi_2\{f(y,\Phi_1)-f(y,\Phi_2)\}\,\mathrm{d}y = 0,$$

whence the monotonicity of f implies that $\Phi_1-\Phi_2$ must have both positive and negative values on $(0,1)$. Thus $h = \Phi_1-\Phi_2$, which vanishes at $y = 0$ and 1, must also vanish at an interior point of the interval $[0,1]$. Further, h satisfies the ordinary differential equation

$$h_{yy}+\theta h = 0, \tag{4.5}$$

in which
$$\theta(y) = \frac{1}{\Phi_1(y)-\Phi_2(y)}\int_{\Phi_2(y)}^{\Phi_1(y)}\left\{f(y,t)+t\frac{\partial}{\partial t}f(y,t)\right\}\mathrm{d}t.$$

Due to the continuity of θ and to the condition (II) satisfied by f, there is a number $\epsilon > 0$ such that $\theta(y) < 4\pi^2-\epsilon$ for $0 \leqslant y \leqslant 1$. Applied to (4.5) and the equation

$$g_{yy}+(4\pi^2-\epsilon)\,g = 0,$$

the sturmian comparison theorem (Coddington & Levinson 1955, ch. 8) shows that any solution to the latter equation must vanish at least twice in the interval $[0,1]$, which conclusion is evidently contradicted by the solution $g = \sin(\sigma y+y_0)$, where $\sigma^2 = 4\pi^2-\epsilon$ and $|y_0| > 0$ is sufficiently small (specifically $2\sigma y_0+y_0^2 < \epsilon$). We are left with the conclusion that Φ is the unique non-trivial x-independent solution of (4.1) and (4.2).

The following attribute of the boundary-value problem (4.1) and (4.2) is noteworthy incidentally because it highlights the appropriateness of the cone to be chosen in §4.1 below. In the case that the monotonicity of f according to condition (III) extends to negative values of its second argument (again see Benjamin 1971, §6), it follows that any bounded solution of the problem tending to zero as $x \to \pm\infty$ is necessarily non-negative on $S = \mathbb{R}\times[0,1]$. Although not claimed as a proof, the following demonstration of this property is simple and plausible. Suppose that ϕ is such a solution of (4.1) and (4.2) that is twice continuously differentiable (see below), satisfies $\phi < 0$ on a subset $\Omega \subset S$ and vanishes on its boundary $\partial\Omega$, which may be in the interior of S or may include all or parts of the lines $y = 0$, $y = 1$ and the asymptotic limits $x \to \pm\infty$ in S. Equation (4.1) for ϕ is now juxtaposed with (4.3) for the first eigenfunction ξ_1, which is well known to be single-signed on $(0,1)$ (see Coddington & Levinson 1955, p. 212). It may thus be specified that $\xi_1(y) > 0$ for $0 < y < 1$. From the equations for ϕ and ξ_1 it follows at once by Green's theorem that

$$\int_\Omega \phi\xi_1\left\{\frac{1}{\lambda_1}f(y,0)-f(y,\phi)\right\}\mathrm{d}x\,\mathrm{d}y = \int_\Omega(\xi_1\,\Delta\phi-\phi\xi_{1yy})\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\partial\Omega}\left(\xi_1\frac{\partial\phi}{\partial n}-\phi\frac{\partial\xi_1}{\partial n}\right)\mathrm{d}s = \int_{\partial\Omega}\xi_1\frac{\partial\phi}{\partial n}\,\mathrm{d}s \geqslant 0.$$

Here the inequality is evident from the positivity of ξ_1 and the fact that, by the definition of Ω, $\partial\phi/\partial n$ cannot have negative values on $\partial\Omega$. But we have $0 < \lambda_1 < 1$ by condition (IV) and $f(y,0) > f(y,\phi)$ if $\phi < 0$ by the extension of condition (III). The first integral over Ω is therefore negative, and a contradiction is established. Note that, according to the same argument, any solution of (4.1) and (4.2) that is periodic in x must also be non-negative. In particular, the x-independent solution Φ is thus confirmed to be so.

To proceed with our main task we recast the problem (4.1) and (4.2) as an integral equation, using the Green function for $-\Delta$ defined by the boundary conditions (4.2). Thus we consider

$$\phi = B[\phi f(\phi)] = A\phi, \quad \text{say}, \tag{4.6}$$

where $f(\phi)$ is a shorthand for $f(y,\phi(x,y))$ and B denotes the symmetric linear operator given by

$$Bu(x,y) = \int_{\mathbb{R}}\int_0^1 k(x-\hat{x},y,\hat{y})\,u(\hat{x},\hat{y})\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}$$

with
$$k(x-\hat{x},y,\hat{y}) = \sum_{n=1}^{\infty}\frac{\exp(-n\pi|x-\hat{x}|)\sin(n\pi y)\sin(n\pi\hat{y})}{n\pi}.$$

The needed properties of B are well known (cf. Amick 1984; Amick & Toland 1989). The kernel k is zero at $y=0,1$ and $\hat{y}=0,1$ but is elsewhere positive and a strictly decreasing function of $|x-\hat{x}|$. Writing $S=\mathbb{R}\times[0,1]$, we have that, qua function of (x,y), $k\in L^p(S)$ for all $p\geqslant 1$ and $(\hat{x},\hat{y})\in S$, and that B is continuous in the senses that it maps bounded elements of $C(S)$ into $C^1(S)$ and bounded elements of $C^\alpha(S)$ into $C^{2,\alpha}(S)$. (If $k=0,1,2,\ldots$, the space $C^k(S)$ is the set of functions in $C(\mathbb{R}\times(0,1))$ whose partial derivatives of order k and less have continuous extensions to S. The space $C^{k,\alpha}(S)$ with $0<\alpha<1$ is the subspace of $C^k(S)$ consisting of functions whose kth-order derivatives are Hölder continuous with exponent α. These spaces have standard norms (cf. Gilbarg & Trudinger 1977).) Accordingly, a solution ϕ of (4.6) that is established in the space $C(S)$ and is bounded can immediately be inferred to be a twice continuously differentiable solution of (4.1) satisfying (4.2). For the condition (I) on f together with the properties of B imply first that $\phi\in C^1(S)\subset C^\alpha(S)$, and then that $\phi\in C^{2,\alpha}(S)$.

4.1. *Choice of Fréchet space and cone*

In this section the theory will be developed for the space $C(S)$ of real continuous functions defined on S, making it a Fréchet space by assigning it the toplogy of uniform convergence on compacta. A suitable metric is again the one given by (2.1), where now the semi-norms p_j are defined by

$$p_j(u) = \max_{\substack{-jl\leqslant x\leqslant jl\\ 0\leqslant y\leqslant 1}} |u(x,y)|, \quad j=1,2,\ldots.$$

As before we shall write $\mathscr{B}_r(v)$ for the open metric ball or radius r centred on a particular element v, and also $\bar{\mathscr{B}}_r(v)$ for its closure.

The cone that will be useful is suggested by the anticipated properties (i) and (ii) of solitary waves. It is

$$K = \{u\in C(S): u(x,y)=u(-x,y)\geqslant 0 \quad \text{and is non-increasing with } |x|\ \forall\, y\in[0,1]\}. \tag{4.7}$$

Evidently this cone is closed and p_1-bounded. Note that the definition of K does not include the boundary conditions (4.2), which are, however, automatically satisfied by any fixed point of

A in K. Note also that the x-independent solution Φ is a fixed point of A in K, for which allowance will have to be made in the same way as for the second trivial solution of the problem in §3. As before, we shall use the notation $K_r = K \cap \mathscr{B}_r(0)$.

4.2. *Properties of the nonlinear operator*

The properties of the operator A in (4.6) now needed are precise counterparts of those established in §3.2 for the corresponding operator in the previous problem. They are presented in three lemmas as follows, the first of which is so close to lemma 3.1 in substance that reference to the earlier proof can be made for most details.

LEMMA 4.1. *A maps K continuously into itself; and, for any $\rho \in (0,1)$, $A(K_\rho)$ is a relatively compact subset of K.*

Proof. Because conditions (I) and (III) imply that $uf(u) \in K$ if $u \in K$, the required property $A(K) \subset K$ will follow from $B(K) \subset K$. This property of B is demonstrable by an immediate adaption of the arguments used in the proof of lemma 3.1. For every $y, \hat{y} \in (0,1)$, the kernel $k(x-\hat{x}, y, \hat{y}) > 0$ is qua function of the variable $z = x - \hat{x}$ (i) a member of $L^1(\mathbb{R})$, (ii) even and (iii) monotonic decreasing on $[0, \infty)$. Hence, as shown before, it is easily verified that $Bu \in K$ if $u \in K$.

To show that A is a continuous self-mapping of K, the argument is virtually the same as before, relying on the fact that continuity is implied by sequential continuity. Thus it suffices to confirm that if any $u \in K$ and any sequence $\{u_n\}_{n=1}^{\infty} \in K$ are such that $p_j(u_n - u) \to 0$ as $n \to \infty$ for each $j = 1, 2, \ldots$, then $p_k(Au_n - Au) \to 0$ as $n \to \infty$ for any given k. By virtue of condition (II) we have that

$$|Au_n - Au| \leqslant 4\pi^2 B(|u_n - u|),$$

and hence the argument proceeds just as was exemplified before in the context of (3.11)–(3.13).

Finally, the relative compactness of $A(K_\rho)$ is established by exactly the same argument that covered the corresponding part of theorem 3.7 (cf. lemmas 3.1 and 3.3). □

LEMMA 4.2. *If $r > 0$ is sufficiently small, then*

$$u \neq tAu \quad \text{for each} \quad u \in K \cap \partial\mathscr{B}_r(0) \quad \text{and} \quad t \in [0,1].$$

Proof. Suppose to the contrary that there is an element u of K with $d(0,u) = r$ and a number $t \in [0,1]$ such that $u = tAu$. For all $(x,y) \in S$, we have $u(x,y) \leqslant u(0,y) = u_0(y)$, say. It therefore follows according to conditions (I) and (III) that

$$u_0(y) = tA(u)(0,y) \geqslant A(u_0)(0,y) = \int_0^1 \tilde{k}(y,\hat{y})\, u_0(\hat{y}) f(\hat{y}, u_0(\hat{y}))\, \mathrm{d}\hat{y}, \tag{4.8}$$

where
$$\tilde{k}(y,\hat{y}) = \int_{\mathbb{R}} k(x,y,\hat{y})\, \mathrm{d}x = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi y)\sin(n\pi\hat{y})}{n^2}$$
$$\left.\begin{aligned} &= \hat{y}(1-y) \quad \text{if} \quad 0 \leqslant \hat{y} \leqslant y, \\ &= y(1-\hat{y}) \quad \text{if} \quad y \leqslant \hat{y} \leqslant 1 \end{aligned}\right\}$$

is the Green function for $-\mathrm{d}^2/\mathrm{d}y^2$ and zero end-conditions at $y = 0, 1$.

Consider now the eigensolution $\xi_1(y)$ of (4.3) corresponding to the first characteristic value, which by condition (IV) can be written $\lambda_1 = 1 - \delta$ with $\delta > 0$. A standard argument regarding the first eigenfunction of any such Sturm–Liouville system shows that $\xi_1(y) > 0$ on $0 < y < 1$ (see Coddington & Levinson 1955, p. 212). We next observe that, because by condition (I) the function f is continuous on $[0, 1] \times [0, \infty)$, a number $\kappa > 0$ certainly exists such that

$$f(s, t) < (1+\tfrac{1}{2}\delta) f(s, 0) \quad \text{for each } s \in [0, 1]$$

if $0 < t \leqslant \kappa$. Hence, in the case that $p_1(u) \leqslant \kappa$, (4.8) leads to

$$0 \leqslant \int_0^1 f(y,0)\, \xi_1(y)\, u_0(y)\, \mathrm{d}y < (1+\tfrac{1}{2}\delta) \int_0^1 \int_0^1 k(y,\hat{y}) f(y,0) f(\hat{y},0)\, \xi_1(y)\, u_0(\hat{y})\, \mathrm{d}y\, \mathrm{d}\hat{y}$$

$$= \lambda_1 (1+\tfrac{1}{2}\delta) \int_0^1 f(\hat{y},0)\, \xi_1(\hat{y})\, u_0(\hat{y})\, \mathrm{d}\hat{y},$$

which is contradictory as $\lambda_1(1+\frac{1}{2}\delta) = 1-\frac{1}{2}\delta-\frac{1}{2}\delta^2 < 1$. Because $p_j(u) = p_1(u)$ for each j if $u \in K$, we have that $p_1(u) \leqslant \kappa$ if $r = d(0, u) \leqslant \kappa/(1+\kappa)$. Thus, for $r \leqslant \kappa/(1+\kappa)$, a contradiction is established, which completes the proof of the lemma. □

Although not needed in the proof, a minor extension of the argument shows that the nonlinear operator A has, at the zero element of $C(S)$, a Fréchet derivative $A'(0)$ with respect to directions into the cone K (see Hamilton 1982 for a general discussion of Fréchet derivatives in Fréchet spaces). This linear operator is specified by

$$A'(0)\, v(x,y) = B[f(y,0)\, v(x,y)], \quad \text{for} \quad v \in K.$$

As was in effect recognized by the proof, $\xi_1 \in K$ is an eigenvector of $A'(0)$, corresponding to the eigenvalue λ_1, and $A'(0)$ has no other eigenvector in K.

In the next lemma we shall refer to the comparable linear operator defined by

$$A'(\infty)\, v(x,y) = B[Q(y)\, v(x,y)],$$

which like A is a compact self-mapping of K. The operator $A'(\infty)$ is in fact the asymptotic derivative of A with respect to the cone K (again see Hamilton 1982), but this fact will not be directly used here. Like the solution ξ_1 of (4.3), the first eigensolution of the Sturm–Liouville problem (4.4) has the property $\eta_1(y) > 0$ on $0 < y < 1$. Thus η_1 is an eigenvector of $A'(\infty)$ in K, corresponding to the characteristic value that by condition (V) can be written $\mu_1 = 1+\iota$ with $\iota > 0$. We shall use the function $H = H(x,y) = h(x)\, \eta_1(y) \in K$, where

$$h(x) = \begin{cases} 1-2(\beta x)^2 + (\beta x)^4 & \text{if} \quad |x| \leqslant \beta^{-1} = b, \\ 0 & \text{if} \quad |x| \geqslant b, \end{cases}$$

and the number $\beta > 0$ will be chosen presently. A calculation shows that

$$\Delta H = \begin{cases} 0 \quad \text{if} \quad |x| < b, \\ -4\beta^2\{1-3(\beta x)^2\}\, \eta_1(y) - \mu_1^{-1} Q(y)\, H(x,y) \\ \qquad \geqslant -(4\beta^2 + \mu_1^{-1} Q)\, H \quad \text{if} \quad |x| \leqslant b, \end{cases}$$

and so it follows that
$$H \leqslant B[(4\beta^2 + \mu_1^{-1} Q)\, H].$$

Taking $\beta > 0$ such that
$$8\beta^2 \leqslant \frac{1}{(1+\iota)\,(1+\frac{1}{2}\iota)} \min_{0 \leqslant y \leqslant 1} Q(y), \tag{4.9}$$

which is evidently possible as Q is a positive continuous function, we have that

$$A'(\infty)\,H = B(QH) \geqslant (1+\tfrac{1}{2}\iota)\,H.$$

This result and the symmetry of B imply that, for any non-zero $u \in K$,

$$\int_{\mathbb{R}}\int_0^1 QH[A'(\infty)\,u-u]\,\mathrm{d}x\,\mathrm{d}y \geqslant \frac{\iota}{2}\int_{\mathbb{R}}\int_0^1 QHu\,\mathrm{d}x\,\mathrm{d}y, \tag{4.10}$$

which is positive because any such u is positive on a subset of

$$\text{support}\,(H) = [-b, b] \times [0, 1]$$

having non-zero measure.

The inequality (4.10) shows that $A'(\infty)\,u-u$ cannot be identically zero on $[-b, b]\times[0,1]$ for any non-zero $u \in K$. Moreover, because b is arbitrary other than being above the least value for which $\beta = 1/b$ satisfies (4.9), this conclusion can be extended to arbitrarily long subsections of S. Let l in the definition of the semi-norms p_j be chosen not less than the least value of b consistent with (4.9). Then (4.10) justifies the stronger statement that, respective to any particular j, there is a number $c = c(j) > 0$ such that

$$p_j(A'(\infty)\,u-u) \geqslant c p_j(u) \quad \text{for each } u \in K. \tag{4.11}$$

For proof, merely suppose (4.11) not to be true, so that a sequence $\{u_n\}_{n=1}^{\infty} \subset K$, $p_j(u_n) = 1$ for all n, can be found with the property that, as $n \to \infty$,

$$A'(\infty)\,u_n - u_n \to 0 \quad \text{on } [-jl, jl] \times [0, 1]. \tag{4.12}$$

As $A'(\infty)$ is a compact mapping of K into itself, either $\{A'(\infty)\,u_n\}_{n=1}^{\infty}$ or some subsequence has a limit $v \in K$, and for (4.12) to hold we must have $p_j(v) = 1$ and $v - A'(\infty)\,v = 0$ on $[-jl, jl]\times[0,1]$. But, with v in place of u and with $b = jl$ in the specification of H, (4.10) plainly gives a contradiction. Thus (4.11) is proved.

We are now in a position to complete the final step in verifying the applicability of the theory developed in §2.

LEMMA 4.3. *If $R < 1$ is sufficiently close to* 1, *then A is homotopic to $A'(\infty)$ on $K \cap \partial\mathscr{B}_R(0)$ and there is no non-zero $u \in K$ such that $u - A'(\infty)\,u \in K$.*

Proof. To prove the first part of the lemma, a contradiction is shown to follow from the supposition that there is a $u \in K \cap \partial\mathscr{B}_R(0)$ and a $t \in [0, 1]$ satisfying

$$u = tAu + (1-t)\,A'(\infty)\,u.$$

For any particular j, this equation implies that

$$p_j(A'(\infty)\,u-u) = tp_j(A'(\infty)\,u - Au) \leqslant p_j(A'(\infty)\,u-Au),$$

and hence (4.11) shows that

$$p_j(A'(\infty)\,u-Au) \geqslant c p_j(u). \tag{4.13}$$

By virtue of condition (V), the quantity $G(y,u) = Q(y) - f(y,u)$ is non-negative for all $y \in [0, 1]$, $u \geqslant 0$; it is non-increasing with u; and for any given $\epsilon > 0$ a number $\sigma > 0$ can be chosen so that, if $u \geqslant \sigma$,

$$G(y,u) < \epsilon \quad \text{for each } y \in [0,1].$$

Writing q for the maximum value of $G(y,0)$ on $[0,1]$ and noting that, because $u \in K$,

$$u(x,y) \leqslant p_1(u) = p_j(u) \quad \text{for each } (x,y) \in S,\ j \geqslant 1,$$

we can hence conclude that

$$0 \leqslant A'(\infty)\,u - Au \leqslant \{\epsilon p_j(u) + q\sigma\}\,B(1) = \tfrac{1}{2}y(1-y)\,\{\epsilon p_j(u) + q\sigma\}$$

and so
$$p_j(A'(\infty)\,u - Au) \leqslant \tfrac{1}{8}\{\epsilon p_j(u) + q\sigma\}. \tag{4.14}$$

The inequalities (4.13) and (4.14) are contradictory if we choose $\epsilon < 4c$ and then require $p_j(u) \geqslant \frac{1}{4}q\sigma/c$, which corresponds to $R \geqslant q\sigma/(4c+q\sigma)$. Thus the first part of the lemma is proved.

The second part of the lemma follows immediately from the inequality (4.10). If there were a non-zero $u \in K$ such that $u - A'(\infty)\,u \in K$, the respective value of the integral on the left-hand side of (4.10) would be non-positive, contrary to the demonstrated positivity of the right-hand side. □

4.3. *Existence theorem*

The preceding three lemmas establish that the theory developed in §2 is applicable to equation (4.6). In particular, lemma 4.2 and lemma 2.6 together imply that $i(K, A, K_r) = 1$ provided $r > 0$ is small enough. Further, the homotopy invariance of the fixed-point index coupled with lemma 4.3 and lemma 2.7 imply that $i(K, A, K_R) = 0$ for $R < 1$ large enough. It is thus guaranteed that the index of A is -1 on the conical segment K_r^R, where $r > 0$ and $R < 1$ are numbers appropriate to the statements of lemmas 4.2 and 4.3, and that accordingly A has a fixed point in K_r^R which would have index -1 if it were unique. To prove a solitary-wave solution, however, we must exclude the x-independent solution Φ from the provisions of the fixed-point theorem. The analysis will proceed on the same lines as in §3.3, showing that if Φ were the only fixed point in K_r^R its index would necessarily be zero. Needless to say, the arguments differ in detail from those in §3.3, but some abbreviation is possible by means of reference to the previous account.

For $u \in K$, let us use the convenient notation

$$F(u) = u(x,y)\,f(y, u(x,y)),$$
$$F'(u) = f(y, u(x,y)) + u(x,y)\,[f_t(y,t)]_{t=u(x,y)}.$$

Thus equation (4.6) may be rewritten $\phi = BF(\phi)$. In particular, the x-independent, non-trivial solution is defined uniquely by $\Phi(y) > 0$ on $(0,1)$ and $\Phi = BF(\Phi) = \tilde{B}F(\Phi)$, where $\tilde{B}$ is the linear operation on x-independent functions that was introduced in (4.8), having the kernel $\tilde{k}(y,\hat{y})$. According to conditions (I) and (III), $F'(\Phi)$ is a continuous, strictly positive function on $[0,1]$; and condition (III) also implies that

$$\Phi F'(\Phi) - F(\Phi) = \Phi^2 f'(\Phi) > 0 \quad \text{on } (0,1). \tag{4.15}$$

We shall need to consider the linear operator $A'(\Phi)$ defined by

$$A'(\Phi)\,v(x,y) = B[F'(\Phi)\,v(x,y)]. \tag{4.16}$$

(This operator is in fact the Fréchet derivative of A at Φ in K (cf. Hamilton 1982), but such an attribution will not need to be used here.) It is easy to confirm (cf. Benjamin 1971, §5) that

$A'(\Phi)$ has an x-independent eigenvector $\zeta_1 \in K$ that satisfies $\zeta_1(y) > 0$ on $(0, 1)$. Thus the equation for ζ_1 is

$$\nu_1 \zeta_1 = \tilde{B}[F'(\Phi)\, \zeta_1],$$

with eigenvalue ν_1. Multiplying this equation by $F(\Phi)$, integrating over $[0, 1]$ and using the symmetry of $\tilde{B}$, we obtain

$$\int_0^1 \zeta_1 \{\nu_1 F(\Phi) - \Phi F'(\Phi)\}\, dy = 0, \tag{4.17}$$

which in view of (4.15) shows that $\nu_1 > 1$. The first of the several results needed to prepare for the existence theorem can now be given.

Lemma 4.4. *If l is chosen sufficiently large, $A'(\Phi)$ has an eigenvector $\zeta_2 = \cos(\pi x/l)\, \bar{\zeta}_2(y)$, with $\bar{\zeta}_2(y) > 0$ on $(0, 1)$, corresponding to which the eigenvalue satisfies $\nu_2 = \nu_2(l) > 1$.*

Proof. According to the definition (4.16) of $A'(\Phi)$, the function $\bar{\zeta}_2(y)$ is required to satisfy

$$\nu_2 \bar{\zeta}_2 = \tilde{B}_l[F'(\Phi)\, \bar{\zeta}_2],$$

where $\tilde{B}_l$ is a symmetric operator akin to $\tilde{B}$ but with the kernel

$$\tilde{k}_l(y, \hat{y}) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(\beta n \pi y) \sin(\beta n \pi \hat{y})}{n^2 + \beta^2} \qquad \left(\beta = \frac{1}{l}\right),$$

$$= \begin{cases} \operatorname{cosech} \beta \sinh(\beta y) \sinh[\beta(1-\hat{y})] & \text{if} \quad 0 \leqslant y \leqslant \hat{y}, \\ \operatorname{cosech} \beta \sinh(\beta \hat{y}) \sinh[\beta(1-y)] & \text{if} \quad \hat{y} \leqslant y \leqslant 1. \end{cases} \tag{4.18}$$

This kernel like $\tilde{k}$ is continuous and positive on $(0, 1) \times (0, 1)$. Hence standard positive-operator methods, exactly as may be used to establish the eigenvector ζ_1 of $A'(\Phi)$, show that $\bar{\zeta}_2$ exists in the class of continuous non-negative functions defined on $[0, 1]$, and the positivity of $\tilde{k}_l$ on $(0, 1) \times (0, 1)$ evidently ensures that $\bar{\zeta}_2 > 0$ on $(0, 1)$.

We note that $\tilde{k}_l \uparrow \tilde{k}$ uniformly on $[0, 1] \times [0, 1]$ as $l \to \infty$ (i.e. $\beta \to 0$), which implies that $\nu_2(l) \to \nu_1$ as $l \to \infty$. It is known from (4.17) that $\nu_1 = 1 + \epsilon$ with $\epsilon > 0$. A finite value l' of l can therefore be found such that $\nu_2(l) \geqslant 1 + \frac{1}{2}\epsilon$ if $l \geqslant l'$. The proof is thus complete. □

The following two, simple results that will be needed are counterparts of lemmas 3.4 and 3.5.

Lemma 4.5. *Let $u(x, y)$ be a continuous function on $[0, l] \times [0, 1]$ that is non-increasing in x but not necessarily positive. Then*

$$\int_0^l \int_0^1 u(x, y) \cos\left(\frac{\pi x}{l}\right) dx\, dy \geqslant 0. \tag{4.19}$$

Equality holds in (4.19) only if $u(x, y) = u(0, y)$ for all $(x, y) \in [0, l] \times [0, 1]$.

Proof. An obvious adaption of the proof of lemma 3.4. □

Lemma 4.6. *Let $x \geqslant l > 0$, $0 < y < 1$, $0 < \hat{y} < 1$. Then, if l is chosen sufficiently large,*

$$\int_0^l \cos\left(\frac{\pi \hat{x}}{l}\right) \{k(x - \hat{x}, y, \hat{y}) + k(x + \hat{x}, y, \hat{y})\}\, d\hat{x} < 0.$$

Proof. From the definition of k below (4.6), this integral is found to equal

$$-\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\{e^{-n\pi(x-l)} - e^{-n\pi(x+l)}\}}{(n^2 + \beta^2)} \sin(n\pi y) \sin(n\pi \hat{y}) = \omega(x, y, \hat{y}) - \omega(x + 2l, y, \hat{y}), \quad \text{say,} \tag{4.20}$$

where, again, $\beta = 1/l$ and $\omega(x,y,\hat{y})$ is a harmonic function of (x,y) on $(l,\infty)\times[0,1]$ for each value of $\hat{y}\in(0,1)$, and so accordingly is the integral that (4.20) represents. We note that $\omega(x,y,\hat{y})$ is zero on $y=0$ and $y=1$, tends to zero uniformly with respect to $y\in(0,1)$ as $x\to\infty$, and on $x=l$ equals $-\frac{1}{2}\tilde{k}_l(y,\hat{y})$ as given by (4.18), being thus negative there for $y\in(0,1)$. By virtue of the maximum principle for harmonic functions, it follows that ω is negative everywhere on $[l,\infty)\times(0,1)$. In further consequence of ω being harmonic on the stated domain, because otherwise its asymptotic property as $x\to\infty$ would be contradicted, a finite value l'' of l must exist such that $\omega(2l,y,\hat{y})>\omega(0,y,\hat{y})$ for all $y,\hat{y}\in(0,1)$ if $l\geqslant l''$. Thus, if $l\geqslant l''$, the harmonic function (4.20) is negative for $x=l$ and $y,\hat{y}\in(0,1)$. Referring the maximum principle to the function (4.20) in this case, we conclude that it is negative on $[l,\infty)\times(0,1)$ if $\hat{y}\in(0,1)$, and thus the lemma is proven. □

As in §3.3, passing reference to the respective periodic-wave problem is needed. We consider the cone

$$K_l=\{u\in C(S): u(-x,y)=u(x,y)\geqslant 0\ \forall(x,y)\in S,\ u \text{ is } 2l\text{-periodic in } x \text{ and non-increasing with } x \text{ on } [0,l]\ \forall y\in[0,1]\},$$

which like K is a closed cone in $C(S)$. As before, the half-period l is identified with the number given the same symbol in the definition of the metric d. Also, l is understood to be large enough to comply with lemmas 4.4–4.6. Again corresponding to steps in §3.3, continuous operators $r_l: K\to K_l$ and $s_l: K_l\to K$ are defined by

$$(r_l u)(x,y)=\begin{cases}u(x,y) & \text{if}\quad 0\leqslant|x|\leqslant l,\\ u(2ml-|x|,y) & \text{if}\quad (2m-1)\,l\leqslant|x|\leqslant(2m+1)\,l\quad(m=1,2,\ldots),\end{cases}$$

$$(s_l v)\,(x,y)=\begin{cases}v(x,y) & \text{if}\quad 0\leqslant|x|\leqslant l,\\ v(l,y) & \text{if}\quad |x|\geqslant l,\end{cases}$$

for all $y\in[0,1]$. Evidently $r_l\circ s_l$ is the same as the identity map on K_l, and so the cone K r_l-dominates K_l.

We shall use the following facts tied to the periodic problem.

LEMMA 4.7. *The operator A maps K_l continuously into itself, and $A(K_l\cap\mathscr{B}_\rho(0))$ with $0<\rho<1$ is a relatively compact subset of K_l.*

Proof. The continuity and compactness of $A: K_l\to K_l$ are confirmable in the same way as in the proof of lemma 4.1. □

It is noteworthy that this lemma, coupled with obvious corollaries of lemmas 4.2 and 4.3, incidentally establishes the applicability of theorem 2.8 to the periodic-wave problem. Moreover, in the proof of lemma 4.8 that follows next, the index of the x-independent fixed point Φ of A is shown in the first place to be zero respecting Φ as a solution of this problem. Thus, all the ingredients of an existence theory for periodic solutions of (4.6) are provided. The available conclusion is that there is an x-dependent solution in every cone K_l with $l>l_c$, where l_c is the value above which l complies with lemma 4.4 (i.e. $\nu_2(l_c)=1$). No direct use needs to be made of this result, but it accords perfectly with the more incisive, generalized view that proposition 2.5 provides regarding the connection between the solitary-wave and periodic-wave problems.

The anticipated result that is the counterpart of lemma 3.6 can now be established.

LEMMA 4.8. *If $\Phi = \Phi(y)$ were the only fixed point of A in K_r^R, where $r > 0$ and $R < 1$ are the numbers specified in lemmas 4.2 and 4.3, the index of this fixed point would be zero, i.e. $i(A, \Phi) = 0$.*

Proof. Considering the homotopy

$$H(u, t) = tAu + (1-t)\, s_l\, Ar_l\, u$$

on

$$K \cap \mathscr{B}_\epsilon(\Phi) = r_l^{-1}[K_l \cap \mathscr{B}_\epsilon(\Phi)], \tag{4.21}$$

with $\epsilon > 0$ sufficiently small, we can retrace exactly the initial steps in the proof of lemma 3.6. The mapping H plainly satisfies the continuity and compactness requirements for an admissible homotopy, thanks to the continuity of r_l and s_l and to the conclusions of lemmas 4.1 and 4.7. Using lemma 4.7, we may therefore conclude that A is homotopic to $s_l\, Ar_l$ on the set (4.21) if there is no $u \in K$ with $d(\Phi, u) = \epsilon$ and no $t \in [0, 1]$ such that

$$u = tAu + (1-t)\, s_l\, Ar_l\, u.$$

The case $t = 1$ is excluded by assumption, and so henceforth it is taken that $t < 1$. For $0 \leqslant x \leqslant l$, this equation reduces to

$$BF(r_l\, u) - u = t\psi, \tag{4.22}$$

where

$$\psi(x, y) = B\{F(r_l\, u) - F(u)\}$$

$$= \int_{\hat{x}>l} \int_0^1 \{k(\hat{x}-x, y, \hat{y}) + k(\hat{x}+x, y, \hat{y})\}\{F(r_l\, u)(\hat{x}, \hat{y}) - F(u)(\hat{x}, \hat{y})\}\, \mathrm{d}\hat{x}\, \mathrm{d}\hat{y}.$$

Multiplying the left-hand side of (4.22) by $F'(\Phi)\, \bar{\zeta}_2(y) \cos(\pi x/l)$, integrating over $[0, l] \times [0, 1]$ and using lemma 4.4, we obtain

$$\int_0^l \int_0^1 \{\nu_2\, F(u) - uF'(\Phi)\}\, \bar{\zeta}_2(y) \cos(\pi x/l)\, \mathrm{d}x\, \mathrm{d}y, \tag{4.23}$$

in which $\nu_2 > 1$. As $u \in K$ is non-increasing with x in $[0, l]$, so also is $\nu_2\, F(u) - uF'(\Phi)$ if

$$\nu_2\, F'(u) - F'(\Phi) > 0;$$

and because $F'(u) > 0$ is a continuous function of u, there is certainly a number $c > 0$ such that this condition is satisfied if

$$p_1(\Phi - u) = \max_{\substack{0 \leqslant x \leqslant l \\ 0 \leqslant y \leqslant 1}} |\Phi(y) - u(x, y)| \leqslant c.$$

Hence, if

$$\epsilon = d(\Phi, u) \leqslant \tfrac{1}{2}c/(1+c), \tag{4.24}$$

the factor in curly brackets in the integrand of (4.23) is non-increasing with x, and so according to lemma 4.5 the value of this integral is non-negative. Moreover, its value is positive unless $u(x, y) = u(0, y)$ for each $(x, y) \in [0, l] \times [0, 1]$.

On the other hand, the corresponding integral derived from the right-hand side of (4.22) is

$$t \int_{\hat{x}>l} \int_0^1 \Gamma(\hat{x}, \hat{y})\, \{F(r_l\, u) - F(u)\}\, \mathrm{d}\hat{x}\, \mathrm{d}\hat{y}, \tag{4.25}$$

in which

$$\Gamma(\hat{x},\hat{y}) = \int_0^l \int_0^1 F'(\Phi)\, \bar{\zeta}_2(\hat{y}) \cos\left(\frac{\pi \hat{x}}{l}\right)\{k(x-\hat{x},y,\hat{y})+k(x+\hat{x},y,\hat{y})\}\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}$$

must be negative according to lemma 4.6.

The only way in which these facts can be reconciled is that (i) $u(x,y) = u(0,y)$ for all $(x,y)\in[0,l]\times[0,1]$ and in addition (ii) either $t=0$ or $u(x,y) = u(l,y)$ for all $x \geqslant l$ and $y \in [0,1]$. If $t = 0$, then u belonging to the set (4.21) satisfies $u = s_l A r_l u$, which with (i) implies that u is x-independent. If the second alternative in (ii) holds, then again (i) implies u to be x-independent. Thus, in either case, we must have $u(x,y) = \psi(y)$, say, a function of y alone. But then we also have $s_l A r_l u = A\psi$ and so $\psi = A\psi$, which contradicts the uniqueness of Φ. It is therefore established that A is homotopic to $s_l A r_l$ on the set (4.21).

Proceeding just as in the proof of lemma 3.6, we note that, because K_l is r_l-dominated by K, therefore

$$i(K, A, r_l^{-1}[K_l \cap \mathscr{B}_\epsilon(\Phi)]) = i(K_l, A, K_l \cap \mathscr{B}_\epsilon(\Phi)); \tag{4.26}$$

and so the proof can be completed by showing the latter index to be zero. Let any $u^* \in K_l$ be chosen that is strictly decreasing with x in $(0,l)$ for $y\in(0,1)$. The aim in view will be met by showing that there is no $u \in K_l$ and no number $a \geqslant 0$ such that

$$u - Au = au^*. \tag{4.27}$$

Supposing the contrary, we may treat (4.27) in the same way as (4.22) to obtain the equation

$$\int_0^l \int_0^1 \{\nu_2 F(u) - uF'(\Phi)\}\, \bar{\zeta}_2(y) \cos\left(\frac{\pi x}{l}\right)\mathrm{d}x\,\mathrm{d}y = -\int_0^l \int_0^1 aF'(\Phi)\, u^* \bar{\zeta}_2(y) \cos\left(\frac{\pi x}{l}\right)\mathrm{d}x\,\mathrm{d}y, \tag{4.28}$$

the right-hand side of which is shown by lemma 4.5 to be non-positive, and strictly negative if $a > 0$. But the left-hand side of (4.27) has already been shown to be non-negative, and strictly positive unless $u(x,y) = u(0,y)$ for all $(x,y)\in[0,l]\times[0,1]$. The only consistent conclusion, therefore, is that $a = 0$ and u obeys the latter restriction. Because $u \in K_l$, we then must have $u(x,y) = \psi(y)$ as before and consequently $\psi = A\psi$, which again contradicts the uniqueness of Φ. It is thus established that

$$i(K_l, A, K_l \cap \mathscr{B}_\epsilon(\Phi)) = 0.$$

In view of the identities (4.21) and (4.26), it follows that

$$i(K, A, r_l^{-1}[K_l \cap \mathscr{B}_\epsilon(\Phi)]) \equiv i(A,\Phi) = 0,$$

and the proof of lemma 4.8 is complete. □

The existence theorem for a solitary-wave solution of (4.6) can now be presented.

THEOREM 4.9. *There exists a* $\phi \in K_r^R$ *satisfying* $\phi = A\phi$ *and*

$$\lim_{|x|\to\infty} \phi(x,y) = 0 \quad \textit{for each} \quad y \in [0,1]. \tag{4.29}$$

Proof. If the x-independent function $\Phi(y)$ were the only fixed point of A in K_r^R, then as remarked at the beginning of §4.3, $i(A,\Phi) = -1$. But lemma 4.8 has established that $i(A,\Phi) = 0$. Therefore a fixed point ϕ is guaranteed that is dependent on x.

To verify (4.29), an argument closely parallel to the one used for the proof of (3.33) in theorem 3.7 can be applied first to show that, because $\phi \in K$, its limit as $|x| \to \infty$ is necessarily either zero or $\Phi(y)$. In the latter case, we would have $v = \phi - \Phi \in K$ and

$$v = B\{F(\Phi + v) - F(\Phi)\} \geqslant B\{F'(\Phi)\, v\}, \tag{4.30}$$

because $F(u)/u = f(y, u)$ is an increasing function of $u \geqslant 0$ for all $y \in [0,1]$. In exactly the same way as the function $H = h(x)\,\eta_1(y) \in K \cap L^1(S)$ was constructed in §4.2. with reference to the property $\mu_1 > 1$, a function $Z = z(x)\,\zeta_1(y) \in K \cap L^1(S)$ can be found satisfying

$$A'(\Phi)\, Z = B\{F'(\Phi)\, Z\} \geqslant (1 + \tfrac{1}{2}\gamma)\, Z,$$

where $\gamma = \nu_1 - 1 > 0$ (see (4.17)). Multiplying (4.30) by $F'(\Phi)\, Z$, integrating over S and using the symmetry of B, we obtain

$$\int_{\mathbb{R}} \int_0^1 F'(\Phi)\, Zv \,\mathrm{d}x\,\mathrm{d}y \geqslant \int_{\mathbb{R}} \int_0^1 B\{F'(\Phi)\, Z\}\, F'(\Phi)\, v \,\mathrm{d}x\,\mathrm{d}y \geqslant (1 + \tfrac{1}{2}\gamma) \int_{\mathbb{R}} \int_0^1 F'(\Phi)\, Zv \,\mathrm{d}x\,\mathrm{d}y,$$

which is contradictory since $F'(\Phi)$, Z and v are all non-zero elements of K, so making the integral positive. This contradiction confirms (4.29), and the proof of the existence theorem is complete. □

Finally, we recall from the discussion after (4.5) that a simple conclusion is available concerning the regularity of solutions. By virtue of the assumed condition (I), the solitary-wave solution of the integral equation (4.5) is simultaneously a regular solution of the partial differential equation (4.1) satisfying the boundary conditions (4.2).

5. Surface solitary waves

We now tackle the classic problem of solitary waves on water in a uniform canal. This problem admits comparatively simple approximate treatments, but the task of demonstrating exact solutions by standard methods is notoriously hard. Our approach is novel and fairly straightforward, again evincing the power of the analytical resources introduced in §2. Exact accounts of surface solitary waves by other means are on the whole more complicated, although they may yield somewhat more information than the non-constructive theory that follows. Notable precedents include the study of Friedrichs & Hyers (1954), in which existence 'in the small' was proved by means of the contraction-mapping principle, and the studies by Amick & Toland (1981 *a*, *b*) in which solitary waves were treated as the limits of periodic waves as their wavelength is taken to infinity. We shall first derive a new form of the Nekrasov integral equation (cf. Milne-Thomson 1968, §14.70) appropriate to solitary waves, and then we shall proceed by verifying a sense in which theorem 2.9 is applicable to it.

5.1. *Derivation of integral equation*

As illustrated in figure 1, a solitary wave is considered as a stationary phenomenon arising on a stream of incompressible perfect fluid with depth h and uniform velocity c at infinity. The origin of the Argand diagram representing the physical plane of the two-dimensional motion is on the canal bottom below the wave crest, and x in $z = x + \mathrm{i}y$ is the horizontal coordinate in the direction of flow. Let $w = \phi + \mathrm{i}\psi$ and let chw denote the complex potential for the

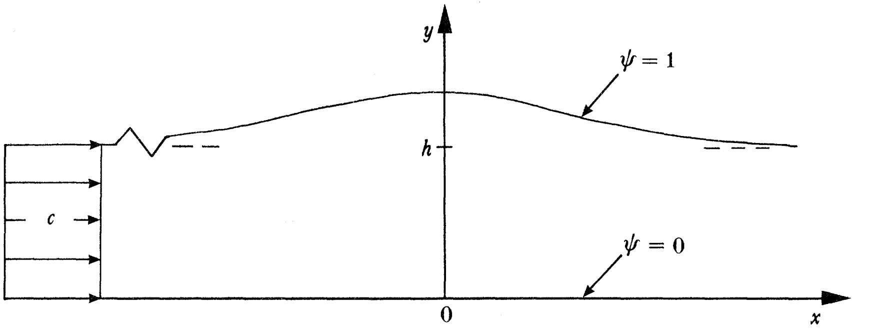

FIGURE 1. Illustration of model for surface solitary wave.

irrotational motion (i.e. $ch\phi$ is the velocity potential and $ch\psi$ is the stream-function), so that the canal bottom is the streamline $\psi = 0$ and the free surface is the streamline $\psi = 1$. The velocity components in the fluid are $(u, v) = ch(\phi_x, \phi_y) = ch(\psi_y, -\psi_x)$, and we write $q^2 = (u^2+v^2)/c^2$.

If the solitary wave of extreme form is excluded from consideration, it can be assumed that $q > 0$ and that z is an analytic function of w in the infinite strip $\mathbb{R} \times [0, 1]$. (For recent theory pertaining to the solitary wave of extreme form, see Amick *et al.* (1982), McLeod (1987) and Amick & Fraenkel (1987).) Hence $h^{-1}\,\mathrm{d}z/\mathrm{d}w = \mathrm{e}^{\mathrm{i}\theta}/q$ is analytic, so is $\ln(h^{-1}\,\mathrm{d}z/\mathrm{d}w) = \ln(1/q) + \mathrm{i}\theta$, and accordingly

$$-(1/q)\,\partial q/\partial\phi = \partial\theta/\partial\psi. \tag{5.1}$$

Here θ is the angle between the x-axis and the tangent to streamlines, so that $\tan\theta = v/u$. As $\theta(\phi, \psi)$ is harmonic in the strip and $\theta(\phi, 0) = 0$, it follows from (5.1) that the function $\theta_s = \theta(\phi, 1)$ is given by

$$\theta_s = -B(q^{-1}\,\partial q/\partial\phi)_{\psi=1}, \tag{5.2}$$

where B is the linear operator defined generally as follows:

$$\left.\begin{array}{l} \text{if} \quad P_{\phi\phi} + P_{\psi\psi} = 0 \quad \text{in} \quad \mathbb{R} \times (0, 1), \\ P(\phi, 0) = 0 \quad \text{and} \quad P_\psi(\phi, 1) = f(\phi) \quad \text{for all} \quad \phi \in \mathbb{R}, \\ \text{then} \quad (Bf)(\phi) = P(\phi, 1). \end{array}\right\} \tag{5.3}$$

Various relevant properties of B will be noted in §5.2 below, including its representation as an integral operator and an appropriate specification of domain and codomain.

At the free surface $\psi = 1$, we also have that

$$(\partial y/\partial\phi)_s = \mathrm{Im}\,(\mathrm{d}z/\mathrm{d}w)_{\psi=1} = h(\sin\theta_s/q_s), \tag{5.4}$$

and the dynamical boundary condition there is that

$$\tfrac{1}{2}(u^2+v^2)_s + gy_s = \text{const.} = \tfrac{1}{2}c^2 + gh,$$

whence

$$c^2 q_s\,\mathrm{d}q_s/\mathrm{d}\phi = -g\,\mathrm{d}y_s/\mathrm{d}\phi. \tag{5.5}$$

Combining (5.2), (5.4) and (5.5), we obtain

$$\theta_s = \mu B\,(\sin\theta_s/q_s^3), \tag{5.6}$$

in which $\mu = gh/c^2$. Further from (5.4) and (5.5), we can find q_s^3 in terms of θ_s by solving

$$\mathrm{d}q_s^3/\mathrm{d}\phi = -3\mu\sin\theta_s. \tag{5.7}$$

As $q \to 1$ as $\phi \to \pm\infty$, an integral with ∞ or $-\infty$ as one limit of integration can at once be written down expressing q_s; but when substituted in (5.6) this expression does not provide a tractable equation for θ_s. The preferable step is to integrate (5.7) from the wave crest, where we can specify that $\phi = 0$ and $q_s(0) = q_0$, say, with $0 < q_0 < 1$. It is also convenient to take $\omega = -\theta_s$ as the dependent variable, so that $\omega(\phi) > 0$ for $\phi > 0$. Thus we obtain

$$q_s^3(\phi) = q_0^3 + 3\mu \int_0^{\phi} \sin\omega(\phi')\, \mathrm{d}\phi' = q_0^3 + 3\mu\phi \int_0^1 \sin\omega(t\phi)\, \mathrm{d}t, \tag{5.8}$$

and hence from (5.6)

$$\omega(\phi) = B\left[\gamma \sin\omega(\phi) \Big/ \left(1 + 3\gamma\phi \int_0^1 \sin\omega(t\phi)\, \mathrm{d}t\right)\right], \tag{5.9}$$

where $\gamma = \mu/q_0^3 = ghc/u_0^3$.

Our analysis will focus on the nonlinear integral equation (5.9), which will be written in short as $\omega = BF_\gamma\omega$. The equation resembles a version of the Nekrasov integral equation that has often been used in the theory of periodic water waves, but its meaning is different, particularly inasmuch as the component linear operator B has been defined to cover solitary waves. Note that γ is the only parameter of the equation. The problem has thus been uncoupled from explicit dependence on the original parameter μ, which characterizes the stream at infinity and so is the more directly significant in physical respects. It is known that $\mu < 1$ for solitary waves, but we shall have no need to use this property. In principle, when a solitary-wave solution of (5.9) is found, the corresponding value of μ can be found from (5.8) in the limit $\phi \to \infty$ which must give $q_s = 1$.

A solitary wave corresponds to a non-trivial solution ω of (5.9) that is an odd function of ϕ vanishing in the limit $\phi \to \infty$. The facts that $|F_\gamma(\omega)| < \gamma|\omega|$ whenever ω is non-zero and that $B(\mathbf{1}) = \mathbf{1}$ (see §5.2) imply (5.9) to have no such solution if $\gamma \leqslant 1$. Accordingly, our aim is to prove that (5.9) has a solitary-wave solution for every number $\gamma > 1$.

Solutions will be established in a space of odd continuous real-valued functions of a real variable, but their further regularity can readily be inferred from the integral equation. A demonstration that each solution is an analytic function generating a good solution of the hydrodynamic problem will be omitted, being essentially straightforward (cf. McLeod 1985). We also pass over the interesting questions posed by the solitary wave of extreme form, which is produced in the limit as $\gamma \to \infty$. Equation (5.9) remains meaningful in this limit (because $\gamma \to \infty$ cancels between the numerator and denominator of the integrand exhibited in (5.9)), but any solution is then necessarily a discontinuous function of ϕ.

5.2. *Properties of B*

For use later, the following properties of the linear operator B defined by (5.3) need to be recognized. We continue to write ϕ for the (dimensionless) independent variable, although its connotation as a velocity potential, is for the moment immaterial.

1. The symbol of B is the even function $\hat{B}: \mathbb{R} \to \mathbb{R}^+$ given by

$$\hat{B}(s) = s^{-1} \tanh(s)$$

(cf. the first paragraph of §3). This fact is plain because, taking $f = \sin(s\phi + \lambda)$ $(s, \lambda \in \mathbb{R})$ in the specification of the boundary-value problem (5.3), we get

$$P = s^{-1} \operatorname{sech}(s) \sin(s\phi + \lambda) \sinh(s\psi)$$

and so
$$Bf = P(\phi, 1) = s^{-1}\tanh(s)\sin(s\phi+\lambda).$$

Remark. The linearized form of (5.9) is $\omega = \mu B\omega$, and evidently $\gamma = \mu$ according to a theory of infinitesimal water waves. Hence the property 1 means that the (dimensional) wavenumber α of simple-harmonic infinitesimal waves satisfies

$$1/\mu = c^2/gh = \tanh(\alpha h)/\alpha h < 1,$$

which recovers the well known dispersion relation between speed c and wavenumber α for such waves.

2. B is equivalent to convolution with the inverse Fourier cosine transform of $\hat{B}(s)$, thus

$$Bf = \int_{\mathbb{R}} k(\phi-\phi')f(\phi')\,\mathrm{d}\phi' \tag{5.10}$$

with
$$k(\phi) = \tfrac{1}{2}\pi\ln\{\operatorname{cotanh}(\tfrac{1}{4}\pi|\phi|)\}. \tag{5.11}$$

Note that
$$\int_{\mathbb{R}} k(\phi)\,\mathrm{d}\phi = 1, \tag{5.12}$$

which fact is obvious from $\hat{B}(0) = 1$, and even more so from the property $B(\boldsymbol{a}) = \boldsymbol{a}$ for all constant functions $\boldsymbol{a}$ that is shown by the specification $P = a\psi$ in (5.5).

3. As is well known, various classes of functions are transformed by B into smoother functions. These properties are deducible by standard arguments from the integral representation (5.10) of B, and also by Sobolev-space arguments referred to the symbol $\hat{B}$. For example, B is continuous from $C_b(\mathbb{R})$ into $C_b^1(\mathbb{R})$, which property bears on our treatment of the integral equation (5.9).

4. It is incidentally noteworthy that, whereas B is most easily appraised as an operation on functions well delimited over the whole of $\mathbb{R}$ (e.g. bounded continuous functions), B still has meaning as an operation on other functions. For example, we have $B\phi = \phi$. This identity could be confirmed by considering the (distributional) Fourier transform of ϕ, multiplying it by $\hat{B}$ and then finding the inverse transform; but the outcome is otherwise obvious from the original definition (5.3) of B and the harmonic function $P = \phi\psi$. Similarly, for $|\beta| < \frac{1}{2}\pi$, we have

$$B(\mathrm{e}^{\beta\phi}) = (\beta^{-1}\tan\beta)\,\mathrm{e}^{\beta\phi}.$$

Remark. Although hardly a proof, the following indication that $\mu < 1$ (i.e. $c^2 > gh$) for solitary waves is simple and interesting. For a solitary wave, $\omega(\phi)\to 0$ as $\phi\to\infty$, and, because the kernel in (5.10) decreases exponentially with distance from the diagonal in $\mathbb{R}\times\mathbb{R}$, it can be expected that
$$|\omega - f|/|\omega| \to 0 \quad \text{as} \quad \phi\to\infty,$$

where f satisfies the linearized form of (5.9) relative to the stream at infinity. That is, $f = \mu Bf$. (A similar inference can, of course, be drawn about the limit $\phi\to-\infty$.) It is easily seen that the only non-zero solution of $f = \mu Bf$ with the property $|f|\to 0$ as $\phi\to\infty$ is

$$f = \text{const.}\times\mathrm{e}^{-\beta\phi} \quad (\beta > 0),$$

with $\mu = \beta/\tan\beta$, which is less than 1 for all $\beta\in(0,\frac{1}{2}\pi)$. (For a similar but rigorous demonstration of the property $\mu < 1$, reference should be made to McLeod (1985).)

5. Consider the set of odd continuous functions

$$\mathscr{K} = \{f\in C(\mathbb{R}) : f(-\phi) = -f(\phi), f(\phi) \geqslant 0 \text{ and } f(\phi)/\phi \text{ is bounded } \forall\phi\geqslant 0\},$$

which evidently satisfies the axioms (2.3) of a cone. In view of the facts (i) that the kernel $k(\phi)$ specified by (5.11) is a positive even function, strictly decreasing with $|\phi|$ increasing, (ii) that every element of $\mathscr{K}$ has bounds $f(\phi) \leqslant a\phi$ for $\phi \geqslant 0$ and $f(\phi) \geqslant a\phi$ for $\phi \leqslant 0$ with a respective positive constant a, and (iii) that $B(a\phi) = a\phi$ as noted in (4), it is easily confirmable that $B(\mathscr{K}) \subseteq \mathscr{K}$.

An important element in the subsequent theory is a subset of $\mathscr{K}$ that is a narrower cone in the Fréchet space $C(\mathbb{R})$. It is

$$K = \{f \in \mathscr{K} : f(s\phi) \geqslant sf(\phi)\ \forall \phi \geqslant 0 \text{ and } s \in [0,1]\}. \tag{5.13}$$

Thus K consists of odd continuous functions that are non-negative and concave with respect to the origin on $[0, \infty)$. The final property discriminating K from $\mathscr{K}$ is equivalently that $f(\phi)/\phi$ is non-increasing on $[0, \infty)$ or, in the case of a continuously differentiable member of K, that $f(\phi) \geqslant \phi f'(\phi)$ on $[0, \phi)$. To confirm that $B(K) \subseteq K$, it remains only to show that Bf has the final property if $f \in K$. Among several proofs that have been noticed by us, the following is the simplest.

Corresponding to any $f \in K$, the harmonic function P defined by (5.3) is necessarily analytic in the interior of the infinite strip, and is an odd function of ϕ for each $\psi \in [0, 1]$. We consider $\zeta = P - \phi P_\phi$. On $\mathbb{R} \times (0, 1)$ this function satisfies the elliptic equation

$$\zeta_{\phi\phi} + \zeta_{\psi\psi} - 2\phi^{-1}\zeta_\phi = 0,$$

to which the maximum principle applies, and we note that $\zeta(\phi, 1) = Bf - \phi(Bf)'$. According to the attribution $f \in K$ and to the definition of P, ζ_ψ cannot be negative as $\psi \uparrow 1$. We also have that $\zeta(\phi, 0) = 0$ for each $\phi \in \mathbb{R}$ and that $\zeta(0, \psi) = 0$ for each $\psi \in [0, 1]$. Hence, referred to the half-strip $(0, \infty) \times (0, 1)$, the maximum principle guarantees that $\zeta(\phi, 1) \geqslant 0$ for all $\phi \geqslant 0$, and so the property in question is proved.

6. Recognizing the Fourier sine transform of $1/\phi$ to be $\frac{1}{2}\pi \operatorname{sgn}(\phi)$, multiplying it by the symbol of B and finding the inverse transform, we conclude that

$$B\left(\frac{1}{\phi}\right) = (\operatorname{sgn}\phi)\left[\tfrac{1}{2}\pi - \mathrm{i}\ln\left\{\frac{\Gamma(\frac{1}{2}+\frac{1}{4}\mathrm{i}|\phi|)\,\Gamma(1-\frac{1}{4}\mathrm{i}|\phi|)}{\Gamma(\frac{1}{2}-\frac{1}{4}\mathrm{i}|\phi|)\,\Gamma(1+\frac{1}{4}\mathrm{i}|\phi|)}\right\}\right] \tag{5.14}$$

(cf. Ditkin & Prudnikov 1965, p. 281). This expression is an odd real function, which has a saltus of π at the origin and is positive for $\phi > 0$ (i.e. its value is $\frac{1}{2}\pi$ at 0^+ and $-\frac{1}{2}\pi$ at 0^-). It is moreover C^∞ and monotonic decreasing for $\phi > 0$, vanishing in the limit $\phi \to \infty$. These properties may be readily confirmed after one notes that, for $\phi > 0$,

$$[B(1/\phi)]' = \psi(\tfrac{1}{2}+\tfrac{1}{4}\mathrm{i}\phi) + \psi(\tfrac{1}{2}-\tfrac{1}{4}\mathrm{i}\phi) - \psi(1+\tfrac{1}{4}\mathrm{i}\phi) - \psi(1-\tfrac{1}{4}\mathrm{i}\phi),$$

where (for the moment only, so with no risk of confusion) ψ denotes the digamma function. The latter expression more evidently than (5.14) is a real function, having negative values for $\phi > 0$.

7. With parameter $a > 0$, a set of functions $\eta_a \in K$ is defined by

$$\eta_a(\phi) = \begin{cases} \sin(\pi\phi/a) & \text{if } 0 \leqslant |\phi| \leqslant a, \\ 0 & \text{if } |\phi| \geqslant 0. \end{cases} \tag{5.15}$$

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As indicated below, it can be shown that for any $\epsilon > 0$, however small, there exists a corresponding positive number a_0 such that, for $a \geqslant a_0$,

$$B(\eta_a) > (1+\epsilon)^{-1}\eta_a \quad \text{if} \quad \phi > 0. \tag{5.16}$$

As η_a and consequently $B(\eta_a)$ are odd functions, the opposite inequality holds, of course, if $\phi < 0$.

To prove this property, the most direct course is to evaluate the representation $B(\eta_a) = \mathscr{F}^{-1}(\hat{B}\mathscr{F}\eta_a)$ by means of contour integration and Jordan's lemma. It is thus fairly easy to obtain

$$B(\eta_a) = \tanh(\pi/a)\,\eta_a/(\pi/a) + \xi_a,$$

in which ξ_a is an odd function found explicitly as an infinite sum of exponential functions, showing that $\xi_a \in \mathscr{K}$ (in fact $\xi_a(\phi) > 0$ if $\phi > 0$). The required property follows at once by virtue of the fact that the continuous function $s^{-1}\tanh s \uparrow 1$ as $s \downarrow 0$.

Needless to say, the special functions η_a, and $B(1/\phi)$ noted in 6, will be used in the analysis that follows.

5.3. *Existence theory*

Introducing the continuous function $M: \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} Mf &= f \quad \text{if} \quad 0 \leqslant |f| \leqslant \tfrac{1}{2}\pi, \\ &= \tfrac{1}{2}\pi \quad \text{if} \quad f \geqslant \tfrac{1}{2}\pi, \\ &= -\tfrac{1}{2}\pi \quad \text{if} \quad f \leqslant -\tfrac{1}{2}\pi, \end{aligned}$$

we modify equation (5.9) in the form

$$\omega = BF_\gamma(M\omega) = A_\gamma\,\omega, \quad \text{say}. \tag{5.17}$$

Any solution of (5.17) satisfying $\sup_{\phi\in\mathbb{R}}|\omega(\phi)| \leqslant \frac{1}{2}\pi$ is a solution of the original equation, but (5.17) offers certain advantages. In particular, proposition 2.5 can be brought to bear directly upon it, illuminating the connection between the solitary-wave problem now to be solved and the underlying periodic-wave problem.

Just as in §3.1, let X denote the Fréchet space based on $C(\mathbb{R})$, with the standard metric as defined by (3.5) and (2.1). As before, we postpone the choice of $l > 0$ fixing the intervals $[-jl, jl]$, $j = 1, 2, \ldots$, on which the semi-norms p_j are defined. The aim in view is to verify that theorem 2.9 is applicable to the operator A_γ for each $\gamma > 1$, proving (5.17) to have a non-trivial solution in the cone $K \subset X$ that has been defined by (5.13).

It needs to be confirmed that A_γ is a continuous positive operator, and is K-compact. First, to show that $A_\gamma(K) \subset K$, we note that if $f \in K$, then obviously $A_\gamma f$ is an odd function and $Af \geqslant 0$ on $[0, \infty)$. Moreover, as f is concave with respect to the origin, so evidently is the function $\sin(Mf)$: that is,

$$\sin\{Mf(s\phi)\} \geqslant \sin\{M(sf(\phi))\} \geqslant s\sin\{Mf(\phi)\}$$

for each $\phi \geqslant 0$ and $s \in [0, 1]$. Hence

$$\begin{aligned} F_\gamma\{Mf(s\phi)\} &\geqslant s\gamma\sin\{Mf(\phi)\} \Big/ \left[1 + 3\gamma s\phi\int_0^1 \sin\{Mf(st\phi)\}\,\mathrm{d}t\right] \\ &\geqslant s\gamma\sin\{Mf(\phi)\} \Big/ \left[1 + 3\gamma\phi\int_0^1 \sin\{Mf(t\phi)\}\right] = sF_\gamma\{Mf(\phi)\} \end{aligned}$$

for each $\phi \geqslant 0$ and $s \in [0, 1]$. Thus we have that $F_\gamma M(K) \subset K$, and it has already been shown in 5 of §5.2 that $B(K) \subseteq K$.

Now, for any $g \in K$, it is plain that

$$\int_0^1 g(t\phi)\,\mathrm{d}t \geqslant \tfrac{1}{2}g(\phi) \quad \text{for all } \phi \geqslant 0. \tag{5.18}$$

Referring this inequality to functions $\sin(Mf) \in K$ appearing above, we conclude directly that, for all $f \in K$,

$$\begin{aligned} F_\gamma Mf(\phi) &\leqslant \tfrac{2}{3}\phi \quad \text{for all } \phi > 0, \\ &\leqslant \tfrac{2}{3}\phi \quad \text{for all } \phi < 0. \end{aligned}$$

It follows that, for all $f \in K$,

$$\left.\begin{aligned} A_\gamma f &\leqslant \tfrac{2}{3}\Omega(\phi) \quad \text{for all } \phi \geqslant 0, \\ &\geqslant \tfrac{2}{3}\Omega(\phi) \quad \text{for all } \phi \leqslant 0, \end{aligned}\right\} \tag{5.19}$$

where $\Omega(\phi) = B(1/\phi)$ is the function exhibited as (5.14). Because $\Omega(\phi)$ is less than $\frac{1}{2}\pi$ and monotonic decreasing for $\phi > 0$, this result shows that

$$\sup_{\substack{f\in K\\ \phi\in\mathbb{R}}} |A_\gamma f(\phi)| \leqslant \tfrac{1}{3}\pi, \tag{5.20}$$

and so $A_\gamma(K) \subset K \cap \mathscr{B}_{\pi/(3+\pi)}$ because by definition $p_1(f) \leqslant p_2(f) \leqslant \dots$. Any solution of (5.17) in K is therefore automatically a solution of the original equation (5.9) without the modification entailing the operation M. The property (5.19) of A_γ amply compensates for the fact that the present cone K is not bounded in the strong sense applicable to the cones used in §§3 and 4 (although it is in fact p_1-bounded in the weaker sense that $p_j(f) \leqslant jp_1(f)$ for every $f \in K$ and $j = 2, 3, \dots$). It may be noted also that the general estimate (5.20) grossly exceeds the maximum slope of solitary waves, which Amick (1987) has proved to be less than $31.15°$; but the estimate is, of course, serviceable in an existence theory.

Verification that $A_\gamma : K \to K$ is continuous and that $A_\gamma(K)$ is a relatively compact subset of K can be established by virtually the same arguments as used in §3.2, and therefore the details are omitted. The crucial facts are (a) that $F_\gamma M : K \to K$ is continuous, (b) that $F_\gamma M(K)$ is contained in the narrow subset of K delimited by (5.19), and (c) that due to the property of B noted in 3 of §5.2, the B-image of this subset is equicontinuous.

The final step required to verify the applicability of theorem 2.9 is presented as follows.

Lemma 5.1. *For every (finite)* $\gamma > 1$, *the operator* A_γ *compresses the cone* K.

Proof. Two conditions will be verified, being considered in the forms (2.12) and (2.14). First, the estimate (5.20) plainly implies that

$$A_\gamma f - f \notin K \quad \text{if } f \in K \cap \mathscr{B}_R,$$

where R is any number such that $1 > R > \pi/(3+\pi)$. Thus the condition (2.12) is satisfied with such a choice of R.

The other condition is a little more difficult to establish. Since $\gamma > 1$, we can choose $\epsilon > 0$ such that

$$\gamma \geqslant 1 + 2\epsilon. \tag{5.21}$$

Next, referring to the definition of the first semi-norm p_1, namely

$$p_1(f) = \max_{0\leqslant|\phi|\leqslant l} |f(\phi)|, \tag{5.22}$$

we choose $l = a$, where $a \geqslant a_0$ provides the property (5.16) in terms of η_a and ϵ. As $f \geqslant \sin Mf \geqslant f - \frac{1}{6}f^3$ for all $f \geqslant 0$, it is seen according to (5.21) and (5.22) that if, for any $f \in K$,

$$3(1+\epsilon)\,\gamma l p_1(f) + \tfrac{1}{6}\gamma p_1^2(f) \leqslant \epsilon, \tag{5.23}$$

then
$$F_\gamma(Mf) \geqslant (1+\epsilon) f \quad \text{on } [0, l], \tag{5.24}$$

with strict inequality at all points where $f(\phi) > 0$. As ϵ and l are already prescribed, there is certainly a number $\delta > 0$ such that (5.23) and consequently (5.24) are satisfied if

$$p_1(f) \leqslant \delta. \tag{5.25}$$

In this case and provided that $p_1(f) > 0$, we have from (5.24) and the property (5.16) that, for any $f \in K$,

$$\int_0^l (f - A_\gamma f)\,\eta_l\,\mathrm{d}\phi \leqslant \int_0^l \{f - (1+\epsilon)\,Bf\}\,\eta_l\,\mathrm{d}\phi = \int_0^l f\{\eta_l - (1+\epsilon)\,B(\eta_l)\} < 0,$$

since B is a symmetric operator on odd continuous functions and $\eta_l(\phi) = 0$ for $\phi \geqslant l$. This result implies that $f - A_\gamma f \notin K$ for all $f \in K$ satisfying (5.25) and $p_1(f) > 0$.

It remains only to confirm that these last two conditions are satisfied on $K \cap \mathscr{B}_r$ for some choice of $r > 0$. As the seminorms p_j are non-decreasing, we have $d(0,f) \geqslant p_1(f)/\{1 + p_1(f)\}$, and so (5.25) is satisfied by choosing $r = \delta/(1+\delta) > 0$. On the other hand, as already noted parenthetically, K is p_1-bounded in the sense that $p_j(f) \leqslant j p_1(f)$ for every $f \in K$. Hence, for all $f \in K \cap \mathscr{B}_r$, we have

$$r = d(0,f) \leqslant \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{j p_1(f)}{\{1 + j p_1(f)\}} < \left(\sum_1^\infty \frac{j}{2^j}\right) p_1(f) = 2p_1(f),$$

which confirms the second condition. Thus, with $r = \delta/(1+\delta)$, the condition (2.14) for compression of the cone K is verified, namely

$$f - A_\gamma f \notin K \quad \text{if} \quad f \in K \cap \mathscr{B}_r,$$

and the proof of the lemma is complete. □

Theorem 5.2. *For each (finite) $\gamma > 1$, equation* (5.9) *has a respective solitary-wave solution ω_γ in the cone K defined by* (5.13). *Each solution satisfies $\omega_\gamma(\phi) \leqslant \frac{2}{3}\Omega(\phi) < \frac{1}{3}\pi$ for all $\phi > 0$, so that $|\omega_\gamma(\phi)| \to 0$ as $\phi \to \pm\infty$.*

Proof. It has been established that, for each γ, A_γ is a continuous positive operator such that $A_\gamma(K)$ is contained in a compact subset of K, and lemma 5.1 has established that A_γ compresses the cone K. Accordingly, theorem 2.9 guarantees that A_γ has a non-trivial fixed point ω_γ in K. The *a priori* estimate (5.19) then shows that ω_γ is also a solution of (5.9). Because $\Omega(\phi) = B(1/\phi)$ given by (5.14) decays to zero like $1/\phi$ as $\phi \to \pm\infty$, the estimate (5.19) also verifies the final statements of the theorem. □

Various other properties of surface solitary waves may be deduced on the basis of the foregoing theory, but we pass over the details. For example, $\omega_\gamma(\phi)$ is positive on $(0, \infty)$, negative on $(-\infty, 0)$. Applied to a solution the estimate (5.19) can easily be improved, for instance by substituting it back into the general γ-dependent bound for $F_\gamma M(K)$ based on (5.18). Continuous dependence of ω_γ on γ can be readily demonstrated at least as a generic property by index-theoretic arguments, but questions concerning uniqueness and the possibility of bifurcation appear to be difficult.

6. Conclusion

The topological method of analysis expounded in §2 has been applied to three specific solitary-wave problems whose variety demonstrates the general effectiveness of the method. A prime fact deserving final re-emphasis is that because the support of solitary-wave solutions in every example is unbounded, standard Banach-space methods are rendered useless for a direct approach. But we should acknowledge the complexities presented in verifying the conditions for applicability of the existence theorems made available in §2.

It is remarkable that the first problem, treated in §3, turns out to be particularly hard, notwithstanding that the one-dimensional system in question may seem superficially an easy case. This problem was in fact treated as the prototype during our studies, and finding how to deal with it conclusively was their watershed. Its principal difficulties, arising from the presence of two separate trivial solutions, were finally well appreciated to be inherent and not avoidable by any elementary means. Also having two extraneous solutions, which represent perfectly legitimate features of the physical model (i.e. a pair of conjugate flows), the problem treated in §4 is comparable with but essentially no more difficult than the prototype. In contrast, because of the neat reduction availed by conformal transformations, the classic water-wave problem treated in §5 avoids the complication of multiple extraneous solutions. Thus, although this problem incidentally presents many interesting details, the gist of the existence theory is comparatively simple.

For the sake of completeness we should point out that counterparts of theorems 2.8 and 2.9, concerning nonlinear operators that expand or compress a Fréchet-space cone K, can be given in terms of Fréchet derivatives with respect to the directions of K (cf. Krasnosel'skii 1964*a*, theorems 4.11 and 4.16). Specifically, one needs to suppose that K is normal and that the continuous nonlinear operator $A: K \to K$ has in this restricted sense a derivative $A'(0)$ at the zero element of K and an asymptotic derivative $A'(\infty)$ in K. If A is K-compact, then so will be these two linear positive operators and they will have eigenvectors in K. Hence statements about the respective eigenvalues in comparison with 1 can replace the conditions (2.12) and (2.13) of theorem 2.8, or (2.15) and (2.16) of theorem 2.9. The essential ideas have been demonstrated in §4, where $A'(0)$ and $A'(\infty)$ were explicitly found as attributes of the particular A under study, and where the verification of the conditions of theorem 2.8 covered virtually the same ground as a proof of the alternative theorem. Because the further details that would need to be spelled out in recalling a proper general definition of Fréchet derivatives of operators in non-normable spaces are not needed for the applications, we have chosen not to include proofs of these other theorems in abstract. We may note, however, that they follow by a straightforward extension of the arguments used in §2.4 on exactly the same lines as in Benjamin (1971, Appendix 1; cf. Fitzpatrick & Petryshyn 1976).

The abstract result stated as proposition 2.5, whose proof is more intricate, was also not strictly needed for the applications. But it was included because of its very general and informative bearing on the connection between periodic and solitary waves. Its relevance to each of the solitary-wave existence theories has been explained at the end of §2.3.

Finally, it should be recognized that this compilation of exact solitary-wave theories is by no means comprehensive. In the same general class many other solitary-wave problems can be listed differing in significant details from the present ones. For example, the theory of solitary waves in a stratified heavy fluid of infinite total depth (Benjamin 1967) has yet to be given a rigorous foundation.

Part of this research was carried out while T.B.B. and J.L.B. were visiting the Mathematics Research Center at the University of Wisconsin, Madison. The final version of the paper was composed while T.B.B. and D.K.B. were visiting the Department of Mathematics at The Pennsylvania State University. We record our indebtedness for the hospitality of these institutions. D.K.B gratefully acknowledges support by the Science and Engineering Research Council for an extended visit to Oxford, during which crucial progress in our investigation was accomplished; and J.L.B. thanks the U.S. National Science Foundation for support. Detailed commentary by two referees has helped significantly to improve the presentation of this paper.

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